UIT Report Vol. 15. No. 2. pp. 385~388. 1984 연구논문집 제15권 제 2 호 pp. 385~388. 1984

Stochastic Motion of a Particle in the Field of Plane Waves

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(Received April 30, 1984)

〈Abstract〉

Stochasticity of nonlinear oscillations is important from many standpoints. The classical motion of a particle in the field of plane waves shows chaotic behaviors for some initial conditions. In this work some aspects of the motion is studied and a local criterion for the stochasticity is suggested which is a preliminary step for next work.

평면파 場 속의 입자의 Stochasticity

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(1984. 4. 30 접수)

〈요 약〉

비선형 진동의 Stochasticity는 여러 점에서 중요하다. 평면파를 중의 입자의 운동에도 어떤 조건하에서 Stochasticity가 나타나는데 여기서는 그 운동의 양상을 고찰한다.

I. Introduction

Stability problems are important in many fields of sciences, in which nonlinear differential equations play their role. There are many situations in physical sciences and biology where the stochastic behaviors of deterministic equations appear.[1] One of these is the motion of a charged particle in the field of two plane waves which is a problem of plasma physics.

The equation of motion of this problem reduces to that of a simple pendulum in the field of a perturbing plane wave. Similar equation holds for noise phenomena in Josephson junctions.[2] Many works have been done on this problem[3],[4] and it has been shown that the seochastic layer can be predicted from semi-quantitive analysis which is called the theory of stochasticity.

The aim of this work and the followings is to study its stochasticity for several cases. In this work, the equation of motion and its stochasticity is studied somewhat qualititively as a preliminary step for next works. At first, the motion in the field of one wave is studied. Then the equation of motion for two-wave case is studied and a local condition for stochasticity is derived for a special case.

II. A Particle in the field of one plane wave.

We consider a charged particle in the field of one plane electromagnetic wave. The equation of motion is,

$$m\ddot{x}=qE_0\sin(kx-wt) \qquad (1)$$

where m, q is the mass and charge of the particle respectively. Changing the variable $(kx-wt)\rightarrow X$, we have

$$\ddot{X}=\frac{kqE_0}{m}\sin X=A\sin X \qquad (2)$$

Eq. (2) is that of unit mass in the periodic potential well $V(X)=A\cos X$ and the analysis can be found in elementary textbooks on mechanics.[5] The position X oscillates in one potential well when initial energy E is less than A or moves with a drift velocity to other wells when E is larger than A. The energy $E=A$ is the separatrix and may be unstable to small perturbations. Then the original position x is,

$$x=\frac{w}{k}t+\text{oscillatory part for } E<A \quad (3)$$

and

$$x=\frac{w}{k}t+\text{monotonic part for } E>A \quad (4)$$

The energy E is given by,

$$E=\frac{1}{2}m|kx_0-w|^2+\frac{kqE_0}{m}\cos kx_0 \qquad (5)$$

and is dependent on initial position, velocity and parameters of the wave field. We notice that the behavior is different especially with different frequencies and this is one of the reason for different behaviors of the layer width between frequency ranges $\nu\gg1$ and $\nu<1$.

Now we consider the motion near one of the separatrix when

$$|kx_0-w|\simeq\sqrt{2A(1-\cos X_0)}. \qquad (6)$$

Near the unstable point $X=0$ the equation of motion reduces to

$$\ddot{X}\simeq AX \qquad (7)$$

and the solution is for $t\ll1$

$$X=C_1e^{\sqrt{A}t}+C_2e^{-\sqrt{A}t} \qquad (8)$$

where constants C_1 and C_2 is determined from the initial condition X_0 and $\dot{X}_0$ When C_1 is positive the particle may flip over to next potential well.

Ⅲ. A particle in the field of two plane waves.

Let us consider a particle with mass m and charge $-e$ in the field of two plane waves. The equation of motion is given by

$$m\frac{d^2x}{d\tau^2}=-eE_1\sin(k_1x-\nu_1\tau)-eE_2\sin(k_2x-\nu_2\tau)-\Gamma\frac{dx}{d\tau} \quad (9)$$

where Γ is the damping constant and τ is the ordinary time.

Changing to a reference frame moving with the phase velocity of the first wave ν_1/k_1 and introducting the variables

$$\tau_0=\sqrt{\frac{m}{k_1eE_1}},\ t=\frac{\tau}{\tau_0} \text{ and } \varphi=k_1x-\nu_1\tau$$

we get the equation of motion in the form

$$\frac{m}{\tau_0^2k_1}\ddot{\varphi}+\Gamma\left(\frac{1}{\tau_0k_1}\dot{\varphi}+\frac{\nu_1}{k_1}\right)+eE_1\sin\varphi=-eE_2\sin\left[\frac{k_2}{k_1}\varphi-\left(\nu_2-\frac{k_2}{k_1}\nu_1\right)\tau_0t\right] \quad (10)$$

where φ is regarded as a function of t. We assume

$$\varepsilon\equiv\left|\frac{E_2}{E_1}\right|=-\frac{E_2}{E_1}\ll1$$

and denote

$$\gamma=\frac{\tau_0}{m}\Gamma,\ \lambda=\frac{k_2}{k_1} \text{ and } \nu=\left(\nu_2-\frac{k_2}{k_1}\nu_1\right)\tau_0.$$

Then we finally get

$$\ddot{\varphi}+\gamma\dot{\varphi}+\sin\varphi=\varepsilon\sin(\lambda\varphi-\nu t) \qquad (11)$$

neglecting the constant term $-\gamma\nu_1\tau_0$ that does not affect resonances.

When there is no dissipation so $\gamma=0$, we have

$$\ddot{\varphi}+\sin\varphi=\varepsilon\sin(\lambda\varphi-\nu t) \qquad (12)$$

Eq. (12) is that of a unit mass in the periodic potential well $V(\varphi)=-\cos\varphi$, which is perturbed by a travelling wave field, that is the cause of the nonlinear resonance. The trajectory of this eqn. always shows periodic motion when initial energy α is lower than some critical energy, α_c which depends on initial position, but when $\alpha>\alpha_c$ near separatrix, it behaves chaotically so that the equation of motion breaks(do wn). Here we make a crude approximation to know what in the equation makes the motion chaotic beside nonlinear resonance effect.

Near one of the unstable points, $\varphi=\pi$, let $x=\varphi-\pi$, then the equation of motion is approximately,

$$\ddot{x}=x-\varepsilon\sin(\nu t-w) \tag{13}$$

where $w=\lambda x+\theta$, and θ is some phase. We assume that w is a constant near the unstable point. When $\nu/\lambda \gg 1$ the solution is

$$x(t)=\frac{\varepsilon}{1+\nu^2}\sin(\nu t-w)+Ae^t+Be^{-t} \tag{14}$$

where

$$A=\frac{1}{2}\left[(x_0+\dot{x}_0)+\frac{\varepsilon}{\sqrt{1+\nu^2}}\sin(w+\phi)\right]$$

$$B=\frac{1}{2}\left[(x_0-\dot{x}_0)+\frac{\varepsilon}{\sqrt{1+\nu^2}}\sin(w-\phi)\right]$$

and

$$\tan\phi=-\nu$$

If we set $x_0=-\sqrt{2(1-\alpha)}$, $\dot{x}_0=0$ where α is the initial energy

$$\alpha=\frac{1}{2}\dot{\varphi}_0^2-\cos\varphi_0=\frac{1}{2}\dot{x}_0^2+\cos x_0. \tag{15}$$

When the factor A before the exponential term is positive, it is possible for the oscillator to flip over to the next potential well, and it is the necessary condition in order for the motion to be stochastic. The condition is

$$1-\alpha<\frac{1}{2}\frac{\varepsilon^2}{1+\nu^2}\sin^2(w+\phi) \tag{16}$$

This condition gives the local and necessary condition for the occurrence of chaos but the resonance is not considered. The energy will fluctuate in the presence of the perturbing field and if resonance happens the condition (16) may be satisfied locally just before the chaotic motion occurs.

Ⅳ. Numerical demonstration of stochastic behaviors.

We solved Eq. (12) numerically and obtained the trajectories. Runge-Kutta and Adams-Moulton methods are adopted to solve the differential equations.

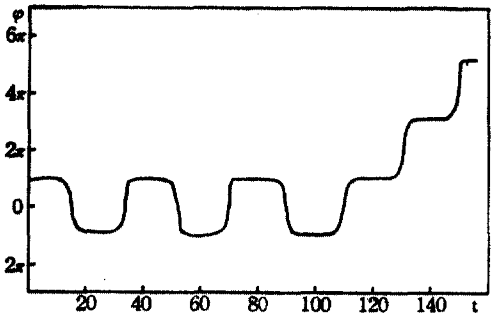

Fig. 1 stochastic behavior when $\nu=10$, $\lambda=0.5$, $\varepsilon=0.01$, $\varphi_0=3.0$, $1-\alpha=0.495\times10^{-5}$

Fig. 1 shows the motion with parameters $\varepsilon=0.01$, $\nu=10$, $\lambda=0.5$ and $\varphi_0=3.0$. When $1-\alpha=0.490\times10^{-5}$ the motion is periodic. But when $1-\alpha=0.495\times10^{-5}$ the chaotic behavior appears after a few periods, although α is lower than before.

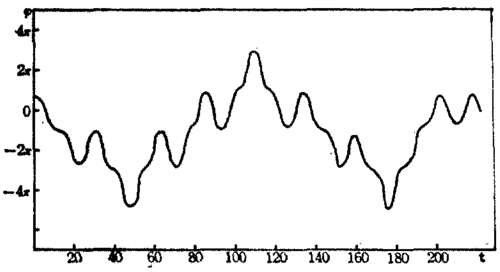

Fig. 2 stochastic behavior when $\nu=0.1$, $\lambda=0.5$, $\varepsilon=0.01$, $\varphi_0=2.5$, $1-\alpha=0.0240$

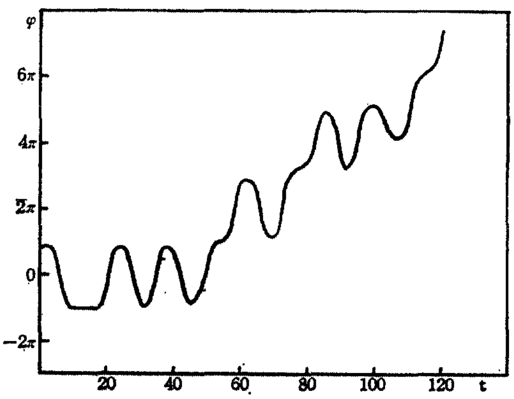

Fig. 3 stochastic behavior when $\nu=0.1$, $\lambda=0.5$, $\varepsilon=0.01$, $\varphi_0=2.5$, $1-\alpha=0.0241$

When $\nu<1$, the chaotic behavior is more remarkable. In Fig. 2 and 3 are shown two different behaviors with slightly different initial energies.

V. Conclusions

We have seen that the motion under the wave fields is highly dependent on initial conditions and parameters of the waves.

We obtained a local criterion for stochasticity. The layer width is decreased with increasing frequency when $\nu/\lambda \gg 1$. The equation of motion is solved numerically to demonstrate the stochastic behaviors.

We wish to express deep gratitudes to Mr. Moo Young Choi for suggesting this work and to Prof. Ohk for kindness during numerical works.

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