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On the Prime and Primary Ideals

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〈Abstract〉

In this paper, we discuss the relationship between a prime ideal and a nil radical of an ideal in a commutative ring with unity and also some properties of primary ideals that a role analogous to the powers of prime number in ordinary arithemetics.

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〈요 약〉

본 논문에서는 여러모로 ideal론에 중요한 역할을 하는 어떤 ideal의 radical과 primary ideal 및 일반적인 계산에서 소수의 거듭제곱과 비슷한 역할을 하는 primary ideal과 prime ideal의 몇가지 성질을 규명하고저 한다.

Ⅰ. Introduction

It is well known that in a nonzero ring with unity maximal ideals always exist. In fact every ideal in the nonzero ring, except the ring itself, is contained in a maximal ideal. And every maximal in a commutative ring with unity is prime ideal.

Using these facts, we show that an ideal and its nil radical are contained in the same prime ideal in the given ring. And we find some properties of primary ideals that play a role analogous to the powers of prime number in ordinary arithmetic.

Ⅱ. Preliminaries

Throughout this paper R denotes a commutative ring with unity.

Definition 1. An ideal I of the ring R is a prime ideal if, for all a, b in R, $ab \in I$ implies that either $a \in I$ or $b \in I$.

Definition 2. Let I be an ideal of the ring R. The nil radical of I, designated by $\sqrt{I}$, is the set $\sqrt{I} = \{r \in R \mid r^n \in I$ for some $n \in Z^+$ (n varies with r)$\}$.

We observe that the nil radical of I may equally well be characterized as the set of elements $r \in R$ whose image $r+I$ in the quotient ring R/I is nilpotent.

Although it is not obvious from the definition $\sqrt{I}$ is actually an ideal of the ring R which contains I. [1]

Definition 3. An ideal I of the ring R is called primary if the conditions $ab \in I$ and $a \notin I$ together imply $b^n \in I$ for some positive integer n.

Clearly, any prime ideal satisfies this definition with $n=1$, and thus, the concept of a primary ideal may be viewed as a natural generlization of a prime ideal.

Notice that an ideal I is primary if $ab\in I$ and $a\notin I$ imply $b\in\sqrt{I}$.

Definition 4. Let I be an ideal of the ring R. A prime ideal P of R is said to be a minimal prime ideal of I(sometimes, an isolated prime ideal of I) if $I\subseteq P$ and there exists no prime ideal P^* of R such that $I\subseteq P^*\subseteq P$.

The following theorems which are essential in this paper are taken from Burton[1], Hungerford[2] Goldhaber[3].

We will state here them without proof.

Theorem(1.1). In a nonzero ring R with unity, maximal ideals always exist. In fact, every ideal in R (except R itself) is contained in a maximal ideal.[2]

Theorem (1.2). In a commutative ring with unity every maximal ideal is a prime ideal.[1),3)]

Ⅲ. Theorems

Theorem(2.1). For any ideal I of R. I and $\sqrt{I}$ are contained in precisely the same prime ideals of R.

Proof. It is enough to show that a prime ideal P(the existence of this P is guaranteed by Theorm(1.1) and Theorm(1.2)) of R that contains I also contains $\sqrt{I}$. For any a in $\sqrt{I}$, there is the smallest positive integer n such that $a^n\in I$.

Since $I\subseteq P$, $a^n\in P$. Then $a^n=a^{n-1}$, $a\in P$ implies $a^{n-1}\in P$ or $a\in P$. If $a\in P$, we've done. Suppose $a^{n-1}\in P$, then $a^{n-1}=a^{n-2}a\in P$.

Therefore we can reduce $a\in P$ by continuing above process. Hence $\sqrt{I}\subseteq P$.

Corollary. If I is a prime ideal of R, then $I=\sqrt{I}$.

Proof. If I is itself a prime ideal, then $I\subseteq\sqrt{I}$ and $\sqrt{I}\subseteq I$ implies $\sqrt{I}=I$.

Theorem(2.2) Let I be a primary ideal of a ring R. Then I has exactly one minimal prime ideal, namely $\sqrt{I}$.

Proof. At first, we prove that $\sqrt{I}$ is a prime ideal. Suppose that $ab\in\sqrt{I}$ and $a\notin\sqrt{I}$. Since $ab\in\sqrt{I}$, there exist $n\in Z^+$ such that $(ab)^n\in I$ i.e, $a^nb^n\in I$, but $a\notin\sqrt{I}$, $a^n\notin I$. Then since I is primary, there is a $m\in Z^+$ such that$(b^n)^m=b^{nm}\in I$. Thus $b\in\sqrt{I}$ and $\sqrt{I}$ is a prime ideal. And by Theorm (2.1), I and $\sqrt{I}$ are contained in precisely the same prime ideal $\sqrt{I}$. And if there is any prime ideal P with $I\subseteq P\subseteq\sqrt{I}$, then again by above Theorm(2,1), $\sqrt{I}\subseteq P$. Hence $P=\sqrt{I}$ and $\sqrt{I}$ is the minimal prime ideal of I.

References

1) David M. Burton, "A First Course in RINGS AND IDEALS" Addison Wesley. 1970, p.77.
2) Thomas W. Hungerford, "Algebra" Holt, Rinehart and Winston. 1974, p.128.
3) Jacob K. Goldhaber and Gertrude Ehrlich, "Algebra" Macmillan, 1970, p.118.