Decomposition of regular openness of sets

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〈Abstract〉

By definitions of a regular open set and a regular semiopen set, we find their properties and show the theorem on the decomposition of a regular open set, that is, a set is regular open if and only if it is open and regular semiopen. Finally, we define new separation axioms and find the implications between them in the part 'application'.

집합의 정칙 개 분할

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〈요 약〉

정칙개집합과 정칙반개집합의 정의에 의하여, 우리는 그들의 제 성질들을 알아보고, 정칙개집합의 분할에 관한정리, 즉 정칙개집합이 되기위한 필요충분조건은 개집합이고 동시에 정칙반개집합임을 보인다. 끝으로 새로운 분리공리들을 정의하여 그들 상호간의 포함관계를 알아본다.

I. Preliminary:

Let A be a subset of a topological space X; $\bar{A}$ and $\mathring{A}$ denote the closure of A and the interior of A respectively. A is said to be regular open if $A=\mathring{\bar{A}}$. Every regular open set is open but not conversely. Intersection of two regular open sets is regular open. A is regular closed if $A=\bar{\mathring{A}}$. Every regular closed set is closed but not conversely. A set is regular open iff its complement is regular closed [2]. A is said to be semiopen if there is an open set O such that $O\subset A\subset \bar{O}$. Every open set is semiopen but not conversely [3]. A space X is said to be semi T_0(resp. semi T_1) if $x, y\in X$, $x\neq y$, there exists a semiopen set U such that $x\in U$, $y\notin U$ or(resp. and) a semiopen set V such that $x\notin V$, $y\in V$ [4]. A space X is semi T_2 if $x, y\in X$, $x\neq y$, there exist disjoint semiopen sets U and V such that $x\in U$, $y\in V$ [4].

II. Decomposition of regular openness of sets

Definitition 1. A subset A of a topological space X is termed regular semiopen if there exists a regular open set O such that $O\subset A\subset \bar{O}$.

Proposition 1: Every regular open set is regular semiopen.

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Proof. Let A be a regular open set, then A itself is a regular open set such that $A\subset A\subset \overline{A}$.

Example 1. Let $X=\{a,b,c\}$ and $T=\{\phi, \{a\}, \{b\}, \{a,b\}, X\}$.

Then the set $\{a,c\}$ is regular semiopen but it is not regular open.

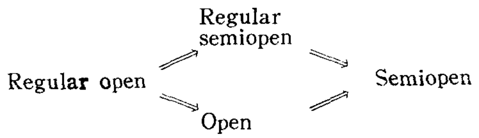
Proposition 2: Every regular semiopen set is semiopen.

Proof. Let A be a regular semiopen set, then there exists a open set O such that $O\subset A\subset \overline{O}$, for a regular open set is open.

Example 2. Let $X=\{a,b,c\}$ and $T=\{\phi, \{a\}, X\}$. Then the set $\{a,b\}$ is semiopen but it is not regular semiopen.

Remark 1. The concepts of regular semiopen and open are independent. For, examine the sets $\{a,c\}$ and $\{a,b\}$ in example 1.

Thus we arrive at the following diagram:



Remark 2. The union of two regular semiopen sets need not be regular semiopen. For, consider the sets $\{a\}$ and $\{b\}$ in example 1.

Remark 3. The intersection two regular semiopen sets need not be regular semiopen. For, consider the sets $\{a,c\}$ and $\{b,c)$ in example 1.

Proposition 3: If A is regular semiopen in X then $\overline{X-A}$ is regular closed.

Proof. Let B be a regular open set such that $B\subset A\subset \overline{B}$. And so, $X-B=\overline{X-B}\supset\overline{X-A}\supset\overline{X-\overline{B}}$ $=\overline{(X-\overline{B})^0}=X-B$. Then, $\overline{X-A}=X-B$.

Corollary 1: If A is regular semiopen then $\dot{A}$ is regular open.

Proof. Let A be regular semiopen, then by proposition 3, $\overline{X-A}$ is regular closed. Since a set is regular open iff its complement is regular closed, $X-\overline{X-A}=\dot{A}$ is regular open.

Corollary 2: If A and B are regular semiopen then $(A\cap B)^0$ is regular open.

Proof. Let A and B be regular semiopen, then by corollary 1, $\dot{A}$ and $\dot{B}$ are regular open. By properties of the interior of a set, $(A\cap B)^0=\dot{A}\cap\dot{B}$. Hence $(A\cap B)^0$ is regular open.

Proposition 4: If A is regular semiopen and $A\subset B\subset\overline{A}$, then B is regular semiopen.

Proof. Let O be a regular open set such that $O\subset A\subset\overline{O}$. And so, $O\subset A\subset B\subset\overline{A}=\overline{O}$.

Proposition 5: If O is open and A regular semiopen $O\cap A$ is semiopen.

Proof. Let B be a regular open set with $B\subset A\subset\overline{B}$. Then for B is open, $O\subset O\subset\overline{O}$ and $O\cap B\subset O\cap A\subset O\cap\overline{B}=\overline{O\cap B}$(cf. Bourbki: General Topology Part I). Accordingly there exists a open set $O\cap B$ such that $O\cap B\subset O\cap A\subset\overline{O\cap B}$. Hence $O\cap A$ is semiopen.

Remark 4. The intersection of an open set and a regular semiopen set may not be regular semiopen. For, in example 1, the set $\{a,b\}$ is open, X is regular semiopen but their intersection $\{a,b\}$ is not regular semiopen. However,

Proposition 6: If O is regular open and A regular semiopen then $O\cap A$ is regular semiopen.

Proof. Let B be a regular open set with $B\subset A\subset\overline{B}$. Then $O\subset O\subset\overline{O}$, $O\cap B\subset O\cap A\subset O\cap\overline{B}\subset \overline{O\cap B}$ since B itself is open. Accordingly there exists a regular open set $O\cap B$ such that $O\cap B \subset O\cap A\subset\overline{O\cap B}$.

Now, from proposition 1 and corollary 1 follows,

Theorem 1: A set is regular opven if and only if it is open and regular semiopen.

Proof. The necessity is obvious from proposition 1.

The sufficiency; Let A be open and regular semiopen. Then since A is open, $A=\dot{A}$, by carollary 1 A is regular open.

The above theorem gives a decomposition of regular openness of sets in view of remark 1.

Theorem 2: Let X and Y be topological paces. If A is regular semiopen in X and B is regular semiopen in Y then $A\times B$ is regular semiopen in $X\times Y$.

Proof. Let U be regular open in X and V

regluar open in Y such that $U\subset A\subset\bar{U}$ and $V\subset B\subset\bar{V}$. And so, $U\times V\subset A\times B\subset\bar{U}\times\bar{V}=\overline{U\times V}$. Moreover, $\overline{(U\times V)}^0=(\bar{U}\times\bar{V})^0=\bar{U}^0\times\bar{V}^0=U\times V$. Therefore, $U\times V$ is regular open in $X\times Y$. Hence $A\times B$ is regular semiopen in $X\times Y$.

Proposition 7: Let Y be a dense subspace of a topological space X and $A\subset Y$. If A is regular open in X then it is regular open in Y.

This is so because it is known [5] that if(Y, T_Y) is a dense subspace of a topogical space(X, T) and $A\subset Y$ then, T_Y-interior (T_Y-closure A) $=\bar{A}^0\cap Y$.

Theorem 3. Let Y be a dense subspace of a topological space X and $A\subset Y$. If A is regular semiopen in X then it is regular semiopen in Y.

Proof. Let O be regular open in X such that $O\subset A\subset\bar{O}$. And so, $O\cap Y\subset A\cap Y\subset\bar{O}\cap Y$. Therefore, $O\subset A\subset$ (closure of O in Y) and by proposition 7, O is regular open in Y.

Ⅲ. Applications

Definition 2. A topological space X is said to be regular semi T_0 if for $x,y\in X$ and $x\neq y$, there exists a regular semiopen set U containing x but not y or containing y but not x.

Remark 5. The seperation axioms of T_0 and regular semi T_0 are independent.

Example 3. Let $X=\{a,b,c,d\}$ and $T=\{\phi, \{a\}, \{b\}, \{a,b\}, X\}$. Then the space X is regular semi T_0 but it is not T_0.

Example 4. Let $X=\{a,b,c\}$ and $T=\{\phi, \{a\}, \{a,c\}, X\}$. Then the space X is T_0 but it is not regular semi T_0.

Proposition 8: Every regular semi T_0 space is semi T_0

Proof. Let X be a regular semi T_0 space and for $x,y\in X$, $x\neq y$. Then there exists a regular semiopen set U such that $x\in U$, $y\notin U$ or $x\notin U$, $y\in U$. Since a regular semiopen set is semiopen, there exists a semiopen set U such that $x\in U$, $y\notin U$ or $x\notin U$ $y\in U$.

Remark 6. The space of example 4 is semi T_0 but it is not regular semi T_0.

Theorem 4: Every regular open dense subspace Y of regular semi T_0 space X is regular semi T_0.

Proof. Let $x,y\in Y$ and $x\neq y$. There exists a regular semiopen set A such that, suppose $x\in A$ and $y\notin A$. By proposition 6, $A\cap Y$ is regular semiopen in X. Since Y is dense in X, by theorem 3, $A\cap Y$ is regular semiopen in Y, and $x\in A\cap Y$ and $y\notin A\cap Y$.

Theorem 5. If X and Y are two regular semi T_0 spaces then so is $X\times Y$.

Proof. Let (x_1,y_1), $(x_2,y_2)\in X\times Y$, such that $(x_1,y_1)\neq(x_2,y_2)$. Suppose that $x_1\neq x_2$. There is a regular semiopen set A in X containing say x_1 but not x_2. Y is regular semiopen in itself. By theorem 2, $A\times Y$ is regular semiopen in $X\times Y$, and $(x_1,y_1)\in A\times Y$ but$(x_2,y_2)\notin A\times Y$.

Definition 3. A topological space X is said to be regular semi T_1 if for $x,y\in X$ and $x\neq y$, there exist regular semiopen sets U and V such that $x\in U$, $y\notin U$ and $x\notin V$, $y\in V$.

Remark 7. The space of example 3 is regular semi T_1 but it is not T_1.

Example 5. An infinite set X equipped with the cofinite topology is T_1 but it is not regular semi T_1, for the only regular semiopen sets are ϕ and X.

Proposition 9: Every regular semi T_1 space is semi T_1.

Proof. Let X be a regular semi T_1 space, then for x, $y\in X$, $x\neq y$, there exist regular semiopen sets U and V such that $x\in U$, $y\notin U$ and $x\notin V$, $y\in V$. Since a regular semiopen set is a semiopen set, the proof is completed.

Remark 8. A semi T_1 space may fail to be regular semi T_1 (example 5).

Proposition 10: Every regular semi T_1 space is regular semi T_0.

Proof. Let X be a regular semi T_1 space and for x, $y\in X$, $x\neq y$. Then there exist regular semiopen sets U and V such that $x\in U$, $y\notin U$

and $x \notin V$, $y \in V$. The above statement implies that there exist a regular semiopen set U containing x but not y or containing y but not x.

Theorem 6 : Every regular open dense subspace of regular semi T_1 space is regular semi T_1.

Proof. Let Y be a regular open dense subspace of a regular semi T_1 space X and for x, $y \in Y$, $x \neq y$. Then there exist regular semiopen sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. Since Y itself is regular open, by proposition 6, the subsets of Y, $U \cap Y$ and $V \cap Y$ are regular semiopen sets in X. Since Y is dense in X, by theorem 3, they are regular semiopen sets in Y such that $x \in U \cap Y$, $y \notin U \cap Y$ and $x \notin V \cap Y$, $y \in V \cap Y$.

Theorem 7 : The product of any two regular semi T_1 space is regular semi T_1.

Proof. Let X and Y be two regular semi T_1 space and for (x_1, y_1), $(x_2, y_2) \in X \times Y$, $(x_1, y_1) \neq (x_2, y_2)$. Assume that $x_1 \neq x_2$. Then there exist regular semiopen sets U and V such that $x_1 \in U$, $x_2 \notin U$ and $x_1 \notin V$, $x_2 \in V$. Since Y itself is a regular semiopen set, by theorem 2, $U \times Y$ and $V \times Y$ are regular semiopen sets in $X \times Y$ such that $(x_1, y_1) \in U \times Y$, $(x_2, y_2) \notin U \times Y$ and $(x_1, y_1) \notin V \times Y$, $(x_2, y_2) \in V \times Y$.

Definition 4. A topological space X is said to be regular semi T_2 if for x, $y \in X$ and $x \neq y$ there exist regular semiopen sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Remark 9. The space of example 3 is regular semi T_2 but it is not T_2.

Proposition 11 : Every regular semi T_2 space is regular semi T_1.

Proof. It is obvious because the condition "disjoint" in regular semi T_2 space need not that in regular semi T_1 space.

Proposition 12 : Every regular semi T_2 space is semi T_2.

Proof. Let X be a regular semi T_2 space and for x, $y \in X$ and $x \neq y$. Then there exist disjoint regular semiopen sets U and V such that $x \in U$, $y \in V$. Since a regular semiopen set is semiopen, the proof is completed.

Theorem 8 : Every regular open dense subspace of regular semi T_2 space is regular semi T_2.

Proof. Let Y be a regular open dense subspace of a regular semi T_2 space X. Then for x, $y \in Y$, $x \neq y$, there exist disjoint regular semiopen sets U and V such that $x \in U$, $y \in V$. Since Y is regular open in itself, by proposition 6, the subsets of Y, $U \cap Y$ and $V \cap Y$ are disjoint regular semiopen sets in X. Since Y is dense in X, by theorem 3, $U \cap Y$ and $V \cap Y$ are disjoint regular semiopen sets in Y such that $x \in U \cap Y$, $y \in V \cap Y$.

Theorem 9 : If X and Y be regular semi T_2 spaces then so is $X \times Y$.

Proof. Let (a, b), $(c, d) \in X \times Y$ and $(a, b) \neq (c, d)$. Suppose that $a \neq c$, $b \neq d$. There exist disjoint regular semiopen sets U, V in X such that $a \in U$, $c \in V$. Similarly, let G, H are disjoint regular semiopen sets in Y such that $b \in G$, $d \in H$. By theorem 2, $U \times G$, $V \times H$ are regular semiopen in $X \times Y$ containing (a, b), (c, d) respectively and $(U \times G) \cap (V \times H) = (U \cap V) \times (G \cap H) = \phi$.

We get the following implication:

$$\begin{array}{ccccc} T_0 & \Longrightarrow & \text{semi } T_0 & \Longleftarrow & \text{regular semi } T_0 \\ \Uparrow & & \Uparrow & & \Uparrow \\ T_1 & \Longrightarrow & \text{semi } T_1 & \Longleftarrow & \text{regular semi } T_1 \\ \Uparrow & & \Uparrow & & \Uparrow \\ T_2 & \Longrightarrow & \text{semi } T_2 & \Longleftarrow & \text{regular semi } T_2 \end{array}$$

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