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Remarks on the Quasi-Projectiveness of Ring

Lee, Kwang Young
Dept. of Basic Studies

〈**Abstract**〉

Let R be a ring with unity $1 \neq 0$, and M a unital right R-module. M is said to be quasi-projective iff any homomorphism $f: M \longrightarrow M/N$ can be lifted to an endomorphism $\bar{f}: M \longrightarrow M$ such that $f = \nu \circ \bar{f}$, where N is a submodule of M and ν is the canonical mapping of M onto the quotient module M/N.

The objective of this note is to find two useful conditions on a ring, namely, a necessary condition and a sufficient condition that every homomorphic image of R be quasi-projective. Also we show that each of the two conditions is not an exact condition for the result with a counter-example respectively.

環의 準射影的性質

이 광 영
기초학과

〈요 약〉

R을 unity $1 \neq 0$을 가진 環이라하고 M을 unital 右 R-module이라 하자. M이 準射影的이라 함은 N을 M의 한 submodule이라 할때 任意의 準同型 寫像 $f: M \longrightarrow M/N$에 대하여 어떤 endomorphism $\bar{f}: M \longrightarrow M$가 있어서 $\nu: M \longrightarrow M/N$을 自然的 準同型對應이라 하면 $f = \nu \circ \bar{f}$가 成立함을 말한다.

우리는 여기서 環 R을 하나의 R-module로 보았을 때 R의 모든 準同型像이 準射影的일 必要條件과 充分條件을 各各 구하고, 또 이들이 完全條件은 될 수 없음을 各各 反例를 들어서 밝혔다.

I. Introduction

A right R-module M is called unital if the ring R has a multiplicative identity 1 and $x1 = x$ for all x in M. Throughout this note, unless otherwise stated, we assume that every ring has unity $1 \neq 0$, and every module is unital. The term "module" without modifier always means right module.

An R-module M is said to be projective in case for any homomorphism $f: M \longrightarrow A$ of the module M into another R-module A and any homomorphism $g: B \longrightarrow A$ of a R-module B onto the module A, there exists a homomorphism $h: M \longrightarrow B$ of the module M into the module B such that the commutativity relation $g \circ h = f$ holds. As a weaker case, quasi-projective module is defined as follows: M is called quasi-projective in case for any R-homomorphism $f: M \longrightarrow M/N$, there is an endomorphism $\bar{f}: M \longrightarrow M$ with the condition $f = \nu \circ \bar{f}$, where N is an R-submodule of M and N is the natural homomorphism of M onto the R-module M/N.

In the definition above, it is immediate that the projectiveness implies the quasi-projectiveness.

Ⅱ. Preliminaries

When M and N are groups, Hom (M, N) denotes the set of all homomorphisms of M into N. Let M and N be right R-modules. By $\mathrm{Hom}_R(M, N)$, we mean the set of all R-homomorphisms of M_R into N_R. Thus $\mathrm{Hom}_R(M, N)=\{f \in \mathrm{Hom}(M, N) \mid f(xr)=f(x)r,\ x \in M, r \in R\}$. If M is a right R-module, M_R, and a left S-module, ${}_SM$, and if $(sx)r=s(xr)$ for all $s \in S$ and $r \in R$, then M is called an (S, R)-bimodule, notationally ${}_SM_R$.

If M is any abelian group, then $E=\mathrm{Hom}(M, M)$ is a ring, called the endomorphism ring of the group M. Here M is a left E-module. If, in addition, M is a right R-module, then $S=\mathrm{Hom}_R(M, M)$ is a subring of E, called the *endomorphism ring* of the module M_R, and so M is an (S, R)-bimodule. Any right ideal I of a ring R can be considered as a right R-module I_R, and any two-sided ideal K is an (R, R)-bimodule ${}_RK_R$.

In general, for any elements $r, s \in R$, ${}_rR$ is a right ideal, R_s is a left ideal, and ${}_rR_s$ is a subring of R. If e is an idempotent element of a ring R, ${}_eR$ is a right ideal of R which is also a left ${}_eR_e$-module.

We can prove that $\mathrm{Hom}_R({}_eR, {}_eR) \cong {}_eR_e$. From this, we see that $\mathrm{Hom}_R(R, R) \cong R$ if R has an identity, and as a matter of fact $\mathrm{Hom}_R(R, R)$ is the set of all left multiplications by the elements of R

A sequence of homomorphisms for modules $A_m \longrightarrow A_{m+1} \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_n$ is said to be exact if, for each $m<k<n$, we have $I_m(A_{k-1} \longrightarrow A_k)=\mathrm{Ker}(A_k \longrightarrow A_{k+1})$. Thus, $A \xrightarrow{f} B$ is monomorphism if and only if $O \longrightarrow A \longrightarrow B$ is exact, and $A \xrightarrow{f} B$ is an epimorphism if and only if $A \longrightarrow B \longrightarrow O$ is exact, where O is the zero module. In particular if the sequence $O \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow O$ is exact, then $A \longrightarrow B$ is a monomorphism, and $B \longrightarrow C$ is an epimorphism with kernel$=I_m(A \longrightarrow B)$. Hence we can see that $C \cong B/I_m(A \rightarrow B)$ in this sequence. An exact sequence of the form $O \longrightarrow A \xrightarrow{f} C \xrightarrow{\pi} O$ is called a short exact sequence. In this case A is isomorphic to $I_m f$, a submodule of B, and C is isomorphic to the factor module $B/I_m f=B/\mathrm{Ker}\,\pi$. Thus if A is a submodule of B, then there is a natural epimorphism $\nu: B \longrightarrow B/A$, and the sequence $O \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\nu} B/A$ is exact, where $\iota: A \longrightarrow B$ is the identity map.

Ⅲ. Discussion and Results

We begin our discussion with an important lemma for projectiveness.

Lemma 1. If R is a ring with unity, then R_R is projective.

A ring R is called right duo if every right ideal of R is two-sided in R. So a commutative ring is clearly a right duo ring.

The following is a well-known property.

Lemma 2. Let R be a ring and the exact sequence $O \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow O$ satisfy the condition that (i) P is projective (ii) N is a $(\mathrm{Hom}_R(P, P), R)$ bisubmodule of P. Then M is quasi-projective.

By this lemma, we can deduce the following characterization.

Proposition 1. If R is a right duo ring, then every homomorphic image of R is quasi-projective.

Proof. Since R is right duo, any right ideal of R is two-sided. Any homomorphic image of R has the form R/I for some right (also two-sided) ideal $I$of R, so we can form the short exact sequence $O \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow O$.

Because I is a two-sided ideal of R, I is a $(\mathrm{Hom}_R(R, R), R)$-bisubmodule of R as a right R-module. We know that R itself is projective as a right R-module R_R. So we conclude, by Lemma 2, that R/I is quasi-projective. Thus the proposition is gained.

However, the converse of this proposition does not hold. We give an counter-example for this as follows:

Example 1. Let F_n be the set of all $n\times n$ matrices over a field F. Then F_n is a ring with unity, and so F_n is projective by Lemma 1. But F_n is not a right duo ring. For the set of all matrices whose entries are all zero except the first row is a right ideal of F_n, but not two-sided in R.

The discussion above tells us that the condition of proposition 1 is sufficient but not necessary.

Now we are going to discuss another part of this section. Three definitions are introduced. A submodule N of an R-module M is called small in M if, for any submodule X of M, $X=M$ whenever $N+X=M$. A module M is said to have a projective cover $P(M)$ if there is an epimorphism $\pi:P(M)\longrightarrow M$ such that $P(M)$ is a projective module and Ker π is small in $P(M)$. The projective cover of a module does not always exist but is unique whenever it exists. A ring R is called right perfect if every right R-module has a projective cover.

Next we state a well-known property.

Lemma 3. If an Rmodule M is quasi-projective and has a projective cover

$$O\longrightarrow \text{Ker }\pi\longrightarrow P(M)\xrightarrow{\pi} M\longrightarrow O,$$

then Ker π is a ($\text{Hom}_R(P(M), P(M))$, R)-bimodule of $P(M)$.

By means of the above lemma, we come to a conclusion.

Proposition 2. If every homomorphic image of a ring R with identity is quasi-proiective, then every small right ideal of R is two-sided.

Proof. Let I be a small right ideal of the ring R and consider the canonical mapping π: $R\longrightarrow R/I$. Then R is a projective cover of an R-module R/I, because the ring R is a projective module by Lemma 1. Since R has a multiplicative identity element, $\text{Hom}_R(R,R)\cong R$, as shown in the section II Preliminaries. Hence $I=\text{Ker }\pi$ is an (R,R)-bisubmodule of R. This implies that the small right ideal of R is two-sided. The proof is complete.

To show the converse of this proposition does not hold, the following is helpful. The Jacobson radical $J(M)$ of an R-module M is defined as the intersection of all maximal submodules of M.

Lemma 4. The Jacobson radical $J(R)$ of a ring R is the sum of all small right ideals of R.

Now a counter-example is presented which enables us to conclude that there is a ring whose small right ideals are all two-sided, but whose epimorphic images are not always quasi-projective.

Example 2. Let Z_2 be the ring of 2×2 matrices over the ring Z of all integers. Since the Jacobson radical $J(Z)$ of the ring Z is O, we see that $J(Z_2)=0$ by the well-know fact that $J(Z_2)=J(Z)_2$, where $J(Z)_2$ is the set of 2×2 matrices over a ring $J(Z)$. Hence, every small right ideal of Z_2 is the O matrix by Lemma 4. This implies that every small right ideal of Z_2 is two-sided.

Now let I be a right ideal of the form

$$\left\{\begin{pmatrix}4m & 4n\\ 0 & 0\end{pmatrix} : m,n\in Z\right\}.$$

and N a right ideal of the form

$$\left\{\begin{pmatrix}4m & 4n\\ a_1 & a_2\end{pmatrix} : a_1, a_2 \text{ arbitrary in } Z\right\}.$$

Then I is an ideal in N.

Consider a mapping $f: R/I\longrightarrow R/N$ such that $f(r+I)=r_0r+N$, where $r_0=\begin{pmatrix}11\\35\end{pmatrix}$. Then we see that the map f is well-defined and Z_2-homomorphism of R/I into R/N. On the other hand, let ν be the natural map of R/I onto $(R/I)/(N/I)\cong R/N$ such that $\nu(r+I)=r+N$. In this case if there exists an Z_2-endomorphism $\bar{f}:R/I\longrightarrow R/I$ with $\nu\circ\bar{f}=f$, we can find a contradiction. This means that R/I is not quasi-projectiv. The counter-example is established.

References

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