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A Note on the T_1-Space and Derived Set

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<Abstract>

A topological space is a T_1-space iff each set which consists of a single point is closed, and the set of all limit points of a set is called the derived set of it.

We show the following: If X is a T_1-space then the derived set of each subset of X is closed, but the converse is not alwys true. Furthermore, as a sharper result, a necessary and sufficient condition that the derived set of each subset be closed is that the derived set of the singleton $\{x\}$ be closed for each point x in a topological space X.

T_1-空間과 導集合에 관한 小考

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<요 약>

位相空間 X가 T_1-space라 함은 X의 각 한점으로 구성된 집합들이 모두 closed일때를 말하고, 한 set의 모든 集積點들의 집합을 導集合이라 부르기로 한다.

우리는 여기서 다음과 같은 사실을 밝힌다. 즉 만약 X가 T_1-空間이면 X의 각 부분집합의 導集合은 closed이지만 그 逆은 성립하지 않으며 더욱 나아가서 어떤 空間 X의 각 부분집합의 導集合이 closed일 필요하고도 충분한 조건은 X의 모든 singleton $\{x\}$의 導集合이 각각 closed인 것이다.

I. Introduction and Preliminaries

In this note, we investigate some properties concerning the limit points and closed sets in topology theory.

We recall, in this section, some necessary definitions and lemmas for the note. We give the propositions and theorem in the next section.

In set theory, the complement of a set A is the set of elements which do not belong to A, that is, the difference of the universal set and the given set A. We denote the complement of A by A^c. The following property is a basic one in set theory.

(Lemma 1). The difference of two sets A and B is equal to the intersection of A and the complement of B, that is, $A-B=A\cap B^c$.

Another important theorem in set theory is due to De Morgan.

(De Morgan's Law).

$$\left(\bigcup_\alpha A_\alpha\right)^c=\bigcap_\alpha A^c_\alpha,\quad \left(\bigcap_\alpha A_\alpha\right)^c=\bigcup_\alpha A^c_\alpha.$$

A topology for a set X is a family $\mathscr{T}$ of subsets of X which satisfies the two conditions;

(i) the union of the members of every subfamily of $\mathscr{T}$ is a member of $\mathscr{T}$, and

(ii) the intersection of any finite members of $\mathscr{T}$ is also a member of $\mathscr{T}$.

In this case, the set X is called the space of the topology $\mathscr{T}$, the pair $(X, \mathscr{T})$ a topological space, and the members of the topology $\mathscr{T}$ are called open relative to $\mathscr{T}$, or $\mathscr{T}$-open, or if only one topology is under consideration, simply open sets. The space X of the topology is always open because X itself is the intersection of the members of the void family of subsits of X, and the empty set ϕ is also open, for it is the union of the members of the void family of the subsets of X.

The open neighborhood of a point x in a topological space X is an open set containing the point x, and is denoted by $N(x)$.

(Lemma 2). A set is open if and only if it contains an open neighborhood of each of its points.

A subset A of a topological space X is called closed iff its relative complement X-A is open. The complement of the complement of a set A is again A, and hence a set is open if and only if its complement is closed. Because, in a topological space, the void set or the whole set is the complement of each other, it is always true that the space and the empty set are closed as well as open. According to De Morgan formulae, the union of a finite number of closed sets is necessarily closed, and the intersection of the members of an arbitrary family of closed sets is closed.

A point x in a topogical space X is a limit point of a subset A of X iff every open neighborhood of x contains at least one point of A other than x itself, that is, $(N(x)-\{x\}) \cap A \neq \phi$.

(Lemma 3). A subset of a topological space is closed if and only if it contains the set of its limit points.

The set of all limit points of a set A is called the derived set of A and denoted by A^d.

(Lemma 4). The union of a set and the derived set of it is closed.

The closure of a subset of a topological space X is defined as the intersection of the members of the family of all closed sets containing the subset. The closure of the set A is denoted by A^-. The set A^- is always closed because it is the intersection of closed sets, and evidently A^- is contained in each closed set which contains A. Consequently A^- is the smallest closed set containing A, and it follows that A is closed if and only if $A=A^-$.

The next lemma describes the closure of a set in terms of its limit points.

(Lemma 5). The closure of any set is the union of the set and its derived set, that is, $A=A \cup A^d$.

Finally we can define the T_1-space as the equivalent form:

(Lemma 6). X is a T_1-space iff for any two points in X they have the open neighborhoods not containing the other of the two points respectively.

II. Propositions and Theorem

(Proposition 1). For any set X, there is a unique smallest topology $\mathscr{T}$ such that $(X, \mathscr{T})$ is a T_1-space.

(Proof). Let $\mathscr{T}$ be the family of all subsets U which are the complements of some finite subsets V of X together with X itself and the void set.

First, for any subfamily $\{U_\alpha\}$ of $\mathscr{T}$, since $\bigcup_\alpha U_\alpha = \bigcup_\alpha V_\alpha{}^c = (\bigcap_\alpha V_\alpha)^c$ by the De Morgan's Law, where each V_α is the complement of U_α respectively, the set $\bigcup_\alpha U_\alpha$ is the complement of a finite subset of X, and it is a member of $\mathscr{T}$.

Second, considering any two members U_1, U_2 of $\mathscr{T}$, since $U_1 \cap U_2 = V_1{}^c \cap V_2{}^c = (V_1 \cup V_2)^c$ by the De Morgan's Law, where V_1 and V_2 are the complements of U_1 and U_2 respectively, the $U_1 \cap U_2$ is the complement of a finite subset of$X$, and it is also a member of $\mathscr{T}$.

Third, for any x in X, $X-\{x\}=\{x\}^c$ is clearly open, and so the singleton $\{x\}$ is closed.

Consequently the family $\mathscr{T}$ is a topology for which $(X, \mathscr{T})$ is a T_1-space.

Last, since the singleton $\{x\}$ is a closed set with respect to any topology $\mathscr{T}'$ for X such that $(X, \mathscr{T}')$ is a T_1-space, every finite set must be closed, and so the complement of it is open. Hence the topology $\mathscr{T}$ is contained in any topology $\mathscr{T}'$ for X such that $(X, \mathscr{T}')$ is a T_1-space, and so $\mathscr{T}$ is the smallest topology satisfying the condition. The proof is complete.

(Proposition 2). If X is a infinite set and $\mathscr{T}$ is the smallest topology for X such that $(X, \mathscr{T})$ is a T_1-space, then $(X, \mathscr{T})$ is connected.

(Proof). From the proposition 1, we know that the smallest topology together with which X is a T_1-space is the very family $\mathscr{T}$ in the above proposition 1 that consists of X itself, the void set, and the subsets which are the complements of some finite subsets of X. By the structure of $\mathscr{T}$, every open set is infinite and every closed set is finite except the void set and the whole set X. Therefore none of subsets of X other than ϕ or X is both open and closed, and this means that $(X, \mathscr{T})$ is connected.

(Proposition 3). If $(X, \mathscr{T})$ is a T_1-space, then the derived set of each subset is closed.

(Proof). Let A be any subset of a topological space X, x a limit point of the derived set A^d, and $N(x)$ an arbitrary open neighborhood of the point x. Then there exists a point y in the set $N(x)\cap A^d$ other than x, that is, y belongs to the set $(N(x)-\{x\})\cap A^d$. Since the singleton $\{x\}$ is closed by……the assumption, $N(x)-\{x\}=N(x)\cap\{x\}^c$ is an open set containing y, and so it is an open neighborhood of y. Because the point y also belongs to A^d by the above stated fact, y is a limit point of A. Hence there exists a point z in the set $(N(x)-\{x\})\cap A$ other than y, too. Thus the set $N(x)$ contains a point in A other than x, and so x is a limit point of A, that is, x is contained in the set A^d. From these facts, A^d contains all of its limit points. Therefore the derived set of each subset is closed by the previous lemma. In the remark at the end of this section, however, we shall see that the converse is not always true.

Now we are going to state the main result that is a sharper one of this proposition 3.

(Theorem). A necessary and sufficient condition that the derived set of each subset be closed in a topological space X is that the derived set of each singleton $\{x\}$ in X be closed.

(Proof). Since every singleton $\{x\}$ is a subset of X respectively, the necessity of the condition is clear. So we have only to show the sufficiency of the condition. Let A be any subset of X. Then we must prove that A^d is closed in X when each $\{x\}^d$ is closed. From the previous lemma, the fact that A^d is closed is equivalent to the fact that it contains all of its limit points, and hence in this case it suffices to show that if x is a limit point of A^d then x is contained in A^d, under the condition that $\{x\}^d$ is closed in X. So we shall prove that if x is a limit point of A^d then x is a limit of A when $\{x\}^d$ is closed in X. Suppose x is a limit point of A^d and $N(x)$ is any open neighborhood of x. Generally a point x can not be a limit point of the singleton $\{x\}$ itself by the definition of the limit point, and so the set $\{x\}^d$ does not contain the point x. Hence the point x belongs to the set $N(x)-\{x\}^d$. In this case the set $N(x)-\{x\}^d=N(x)\cap(\{x\}^d)^c$ is open in X, for $\{x\}^d$ is closed by the assumption, and so $N(x)-\{x\}^d$ is an open neighborhood of the point x. Since x is a limit point of A^d, there exists a point y in the set $((N(x)-\{x\}^d)-\{x\})\cap A^d$.

Now, from the previous lemmas and set theory, we have the following identities:

$$\begin{aligned}(N(x)-\{x\}^d)-\{x\}&=(N(x)\cap(\{x\}^d)^c)\cap\{x\}^c\\&=N(x)\cap((\{x\}^d)^c\cap\{x\}^c)=N(x)\cap(\{x\}^d\cup\{x\})^c\\&=N(x)\cap(\{x\}^-)^c.\end{aligned}$$

And the set $\{x\}^-$ is closed by the lemma that the closure of any set is closed. So the set

$(N(x)-\{x\}^d)-\{x\}$ is an open set, furthermore, an open neighborhood of y, for y belongs to the set $(N(x)-\{x\}^d)-\{x\}$ by the above mentioned fact. Also, since y is a limit point of A from the fact that y belongs to the set $[(N(x)-\{x\}^d)-\{x\}]\cap A^d$ above and hence particularly belongs to A^d, there exists a point z in the set:

$[((N(x)-\{x\}^d)-\{x\})-\{y\}]\cap A=((N(x)-\{x\}^-)-\{y\})\cap A$ by the nature of A^d. Because the point z belongs to the set $(N(x)-\{x\}^-)-\{y\}$ and clearly the set $(N(x)-\{x\}^-)-\{y\}$ is contained in the set $N(x)-\{x\}$ by the fact that $\{x\}^-$ contains $\{x\}$, the point z necessarily belongs to the set $N(x)-\{x\}$, and z is a pint of A from the above fact that z belongs to the set $((N(x)-\{x\}^-)-\{y\})\cap A$. Accordingly $N(x)$ contains a point z of A other than x, that is, x is a limit point of A. Thus the theorem is established.

(Remark). If a singleton$\{x\}$ is closed in a topological space X, the derived set $\{x\}^d$ is closed, but the converse does not hold. We prove this fact. Suppose the set $\{x\}$ is closed. Then$\{x\}^c$ is open, and the singleton $\{x\}$ must contain all of its limit points in it. But since it is always true that the point x itself can not be a limit point of the singleton $\{x\}$ by the definition of limit point, $\{x\}^d$ must be the void set, and so, $\{x\}^d$ is closed in X by the nature of topology. To show that the converse is not always true, we give a counter-example as follows; Let X be the set $\{a, b\}$, and $\mathscr{T}$ the family of subsets of X such that $\mathscr{T}=\{\phi, \{a\}, \{a, b\}\}$. Then we can check that $\mathscr{T}$ forms a topology for X. In this case the singleton $\{a\}$ is open in $(X, \mathscr{T})$, but not closed for $\{a\}^c=\{b\}$ is not open in this topological space. Now consider the derived set $\{a\}^d$ of the singleton $\{a\}$. Because evidently $\{a\}^d$ is $\{b\}$, and $\{b\}=\{a\}^c$ is closed, the derived set $\{a\}^d$ is closed in the topological space $\{X, \mathscr{T}\}$. But the set $\{a\}$ itself is not closed in X by the above statement. From this, we see that the converse is not always true. Of course, in this case we can also deduce that the converse of the Proposition 3 does not hold by the theorem of this section. So, we can find the following: The condition that the derived set of each singleton be closed is coarser than that of T_1-space.

References

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