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Derivation of the Tensor-Products of C^*-algebras

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〈Abstract〉

Let A and B be C^*-algebras. If A or B has outer derivations, $A \otimes B$ has outer derivations. Among the case of A and B with only inner derivations, we are going to show that if A is a Von Neuman algebra and B is a commutative algebra, then every derivation of $A \otimes B$ is inner.

C^*-대수들의 텐서적에서의 미분

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〈요 약〉

$A \otimes B$ 가 C^*-대수일 때, A 혹은 B 가 외부미분(outer derivation)을 갖으면 $A \otimes B$ 는 외부 미분을 갖는다. 이 논문에서는 A, B 가 모두 내부미분(inner derivation)만을 갖는 경우 중 A 가 Von Neumann 대수이고 B 가 가환 C^*-대수일 때 $A \otimes B$ 가 내부미분만을 갖음을 보였다.

I. Introduction

A derivation of a C^*-algebras A is a linear map D on A satisfying $D(ab)=D(a)b+aD(b)$ for all $a, b \in A$. Every derivation of A is norm continuous ([1]) and ultra weakly continuous in any representation of the C^*-algebra. Each norm continuous one parameter group of automorphism of A is of the form

$$A \ni a \cdots\cdots \rightarrow \exp(tD)(a)$$

where D is a derivation of A.

If there exists an element $b \in A$ that $D(a)=ab-ba$ for all $a \in A$, then D is called an inner derivation. The question of which C^*-algebras have only inner derivation has been considered by many authors. Sakai showed that W^*-algebra, uniformly hyperfinite C^*-algebra and simple C^*-algebras with unit have only inner derivations ([5], [6], [7]). The case of separable C^*-algebra has been completely by G. A. Elliot, Pederson and Kadison ([2], [3]).

This note represents the derivation of the C^*-tensor product of C^*-algebras.

II. Derivations on the von Neumann algebras

Let $L(A)$ be the Banach space of the bonuded linear operators on a C^*-algebra A and $\Delta(A)$ be the space of derivations on A. Hence $\Delta(A)$ is the subset of L(A).

Definition 2.1. Let D_x be the inner derivation of A. We define the norm of D_x as follows;

$$\|D_x\| = \sup\{\|xa-ax\| : a \in A \text{ and } \|a\|=1\}.$$

Lemma 2.2. Let A be a von Neumann algebra and X a point-norm compact subset of $\Delta(A)$. Then X is norm compact in $L(A)$.

Proof. Suppose that A has a von Neumann subalgebra invariant under X with a countable weak* dense subset. We assume that X is not norm compact. By the point-norm compactness of X, the pointwise norm sequence of X is uniformly convergent. Hence X is norm closed in $L(A)$ and by the completeness of $L(A)$, it is complete. Since X is not norm compact, X can't be totally bounded. So there exist an $\varepsilon>0$ and a sequence of derivations $\{D_i\}\subset X$ such that $\|D_i-D_j\|>\varepsilon$ for all $i\neq j$. We may choose an element $a_{ij}\in A$ satisfying $\|a_{ij}\|<1$ and $\|(D_i-D_j)a_{ij}\|>\varepsilon$ for each i, $j=1, 2, \cdots(i\neq j)$. Let C be the *-algebra over the rationals of all polynomials in the variables $\{D_n{}^k(a_{ij})\}$, where $i, j, n=1, 2, \cdots(i\neq j)$ and $k=0, 1 \cdots$. Cleary, C is countable and invariant under each D_j. Let $\tilde{C}$ be the von Neumann subalgebra of A generated by C. Then $D_j(\tilde{C})\subset\tilde{C}$ for all D_j and $\tilde{C}$ has countable weak* dense subset. This contradicts $\|(D_i-D_j)a_{ij}\|>\varepsilon$ for all $i\neq j$.

Theorem 2.3. Let A be a von Neumann algebra and B be a commutative C^* algebra with unit. Then $A\otimes B$ has only inner derivations.

Proof. We may regard $A\otimes B$ as $C(\Omega, A)$, the space of continuous A-valued function on the spectrum space Ω of B. Let D be a derivation of $A\otimes B$. It determines a derivation $D_w(a)=D(\alpha)(w)$, where α is the constant function in $C(\Omega, A)$ whose value is a. We consider the map $\theta: w\to D_w$ on Ω. By the norm coninuity of derivations, θ is continuous for the point-norm topology on $L(A)$. Sinc Ω is compact, $\{D_w\}_{w\in\Omega}$ is point norm compact. And by Lemma 2.2, $\{D_w\}_{w\in\Omega}$ is norm compact, so point-norm and norm topology agree on $\{D_w\}_{w\in\Omega}$.

Futher, we represent the map $\Phi: a\longrightarrow ad(a)$ from $A/Z(A)$ to $\Delta(A)$, whose $Z(A)$ is the center of A. Since every derivation of von Neumann algebra is inner ([4]), it is onto. By [8, theorem 5] Φ is bicontinuous, so it is homeomorphism. Hence $\Phi^{-1}\theta$ is a continuous map Ψ on Ω to $A/Z(A)$. By selection theorem [3], Ψ has a selection Γ in $C(\Omega, A)$ and cleary $D=ad(\Gamma)$.

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