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Integrality and Jacobson Radicals of Finite Normalizing Extension Rings

Lee, Dong-Su
Dept. of Mathematics
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〈Abstract〉

In this paper we consider finite normlizing extension rings. We will generalize the integrality results which are introduced by Martin Lorenz. Using this results we show that the Jacobson Radicals of R and S are related by $J(S)\cap R$ where S is a finite normalizing extension ring of R.

유한정규 확장환에 있어 Integrality와 Jacobson Radical

이 동 수
수 학 과
(1981. 12. 30 접수)

〈요 약〉

본 논문에서는 유한정규 확장환에 있어 몇가지 성질을 다루도록 하겠다. Martin Lorenz와 D. S. Passman에 의해 소개되어진 환에 있어서의 Integrality성질을 보다 일반화 시키고 그 결과를 이용해 R의 Jacobson Radical J(R)과 S의 Jacobson Radical $J(S)$사이의 관계가 $J(R)=J(S)\cap R$ 이라는 것을 보여 주었다.

I. Introduction.

Let S be a ring and R be a subring with the same 1. The extension $R\subset S$ is called a finite normalizing extension if there exist finitely many elements $x_1, x_2, \cdots x_n$ in $S \sum_{i=1}^{n} Rx_i$ and $x_iR=Rx_i$ for all i. This extension theory is studied at first by Edward Formanek and Arun Vinayak Jategaonkar [3]. Martin Lorenz showed that S is over R with normal ideal in the sense of Schelter [7, 8]. Here normal ideal A means that A is an ideal of S and $Ax_i=x_iA$ for all i. In this paper we will treat general ideal of S instead of normal ideal.

II. Integrality.

We define some terminologies which are defined by Martin Lorenz in some different method.

Definition 2.1. Let $n\geqq 1$ be a positive integer and let $A=(A_1, A_2, \cdots A_n)$ be a fixed seqence of n right ideals A_i of R. A metrix $\alpha \in M_n(R)$ is called an *A-matrix* if and only if all entries in the i-th row of α belong to A_i.

Definition 2.2. Let D_A denote the subring of $M_n(R)$ consisting of those A-matrices which are diagonal, so that $D_A \simeq A_1 \oplus A_2 \oplus \cdots\cdots \oplus A_n$. A subring $T\subset D_A$ is called an *A-transversal* if and only if T projects onto each component A_i.

We give a very important example of these definitions. Let S be a finite normalizing extension of R and A be a right ideal of R. For $i=1,2,\cdots n$ define

$$A_i=\{r \varepsilon R \mid rx_i \varepsilon x_i A\}$$

Since $A_iRx_i=A_ix_iR=x_iAR\subset x_iA$, A_i is a right ideal of R. Let $A=(A_1,A_2\cdots A_n)$ and let T_A denote the set of all diagonal matrices $\alpha=\text{diag}(r_1,r_2,\cdots r_n) \varepsilon M_n(R)$ such that there exists an element $a \varepsilon A$ with $r_ix_i=x_ia$ for $i=1,2,\cdots n$. Then the elements of T_A are clearly A-matrices and T_A is even A-transversal. For, if $r \varepsilon A_i$ is given then we can choose $a \varepsilon A$ with $rx_i=x_ia$, and then suitable $r_j \varepsilon A_j$ with $r_jx_j=x_ja$ $(j\neq i)$. Clearly, $\alpha=\text{diag}(r_1,..r_{i-1},r,r_{i+1},\cdots r_n)$ belong to T_A, and α maps to r under the projective map of T_A to A_i. Thus T_A maps onto each A_i, and it is easily cheked that T_A is a subring of D_A.

Definition 2.3. Let $A=(A_1,A_2,\cdots A_n)$ be a fixed n-tuple of right ideals of R and let $T\subset D_A$ be an A-transversal Let $\alpha_1,\alpha_2,\ldots\ldots\alpha_m$ be A-matrices. By a T-monomial in α_1, $\alpha_2,\cdots\alpha_m$ we will mean a product of the form

$$d_1\alpha_{i_1}d_2\alpha_{i_2}\ldots\alpha_{i_r}d_{r+1}.$$

where the d_i's belong to T and where some of the d_i's may be missing but at least one d_j does occur. For example, if d_1 and d_2 belong to T then $d_1\alpha_1\alpha_2d_2\alpha_1$ and d_1 are T-monomials in α_1 and α_2, but $\alpha_1\alpha_2$ is not T-monomial. Any sum of T-monomials will be called a T-polynomial, and its degree will be the highest degree of the monomials occuring in the sum where the degree of such a monomial is the total degree in the α_i's.

From these definions Martin Lorenz proved the following Proposition [7].

Propsition 2.4. *Let R be a ring and let $M_n(R)$ be the ring of n by n matrices over R. Then there exists an integer $t=t(n)\geqq 1$ such that the following holds:*

Let $A=(A_1,A_2,\ldots\ldots A_n)$ be a fixed n-tuple of right ideals of R and let $T\subset D_A$ be an A-transvesal. If $\alpha_1,\alpha_2,\cdots\alpha_t \varepsilon M_n(R)$ are arbitary A-matrices then

$$\alpha_1\alpha_2\cdots\alpha_t=\Psi(\alpha_1,\alpha_2,\alpha_3,\cdots\alpha_t)$$

for a suitable T-polynomial Ψ $(\alpha_1,\alpha_2,\cdots\alpha_t)$ of degree at most t-1.

proof. See [7].

From this proposition we get the following theorem.

Theorem 2.5. (Generalization of Martin Loren's result) *Les S be a finite normalizing extension ring of a ring R. Then there exists an integer $t\geqq 1$, depending only upon n, such that for any right ideal A of R and any $s_1,s_2,\cdots s_t \varepsilon AS$ we have $s_1s_2\cdots s_t=\Psi(s_1,s_2,\cdots s_t)$ where $\Psi(s_1,\cdots s_t)$ is an A-polynomial of degree less than t.*

proff. Set $F=R\oplus R\oplus\cdot\oplus R$ (n times) so that F is a free left R-module of rank n, and let $h:F\rightarrow S$ be the left R-module homomorphism given by

$$h(r_1,r_2,\cdots r_n)=\sum_{i=1}^{n} r_ix_i.$$

If K is the kernel of h, then $E=\{\sigma \varepsilon End_RF \mid \sigma(K)\subset K\}$ is a subring of End_RF where $End_R F$ denotes the set of all endomorphism from F into itself. On the other hand we know that End_RF is isomorpic to $M_n(R)$. In this case we define an endomorphism $\hat{\sigma} \varepsilon End_RS$ which is given by $\hat{\sigma}(s)=h\hat{\sigma}(r_1, r_2, \cdots r_n)$ Since $\sum r_ix_i=0$ then $\hat{\sigma}(s)=0$, $\hat{\sigma}$ is well difined and this is an endomorphism of S.

Now let $t=t(n)$ be the integer given by proposition 2.4. and let $s_1,\cdots s_t \varepsilon AS$ be given, with $s_k=\sum_{j=1}^{n} a_j^{(k)}x_j$, $a_j^{(k)} \varepsilon A$. Then $x\,a^{(k)}{}_j=b_{ij}^{(k)}x_j$ for suitable elements $b_{ij}^{(k)} \varepsilon A_i=\{r \varepsilon R \mid rx_i \varepsilon x_i A\}$ and hence

$$x_is_k=\sum_{j=1}^{n}x_ia_j^{(k)}x_j=\sum_{j=1}^{n}b_{ij}^{(k)}x_ix_j=\sum_{j=1}^{n}r^{(k)}{}_{ij}x_j$$

for suitable $r_{ij}^{(k)} \varepsilon R$. Since $b_{ij}^{(k)} \varepsilon A$. and A_i is a right ideal of R, $r_{ij}^{(k)} \varepsilon A_i$. Let $\alpha_k=(r_{ij}^{(k)}) \varepsilon M_n(R)$ for $k=1,2,\cdots t$. and set $A=(A_1,A_2,\cdots$

A_n). Then each α_k is an A-matrix. Moreover, for any $f=(r_1, r_1, \cdots r_n) \varepsilon F$ we have the followings

$$h(f\alpha_k)=h(\sum_{i=1}^{n} r_i r_{i1}^{(k)}, \sum_{i=1}^{n} r_i r_{i2}^{(k)} \cdots \sum_{h=1}^{n} r_i r_{in}^{(k)})$$
$$= \sum_{j=1}^{n}(\sum_{i=1}^{n} r_i r_{ij}^{(k)})x_j$$
$$= \sum_{i=1}^{n} r_i \sum_{j=1}^{n} r_{ij}^{(k)} x_j$$
$$=(\sum_{i=1}^{n} r_i x_i)s_k$$
$$=h(f)s_k$$

Hence we see that $\alpha_1 \varepsilon E$ and that the endomorphism $\hat{\alpha}_1$ of S is right multiplication by S_1.

Let T_A denotes the set of all diagonal matrices $d=(r_1, \cdots r_r) \varepsilon M_r(R)$ such that $r_i x_i = x_i a$ for a suitable $a \varepsilon A$. As we have seen earlier, T_A is an A-transversal in $M_n(R)$. Moreover, as in the preceding paragraph, we see that if $d \varepsilon T_A$ then for all $f \varepsilon F$ we have

$$h(fd)=h(f)a.$$

for a suitable $a \varepsilon E$ and d is right multiplication by $a \varepsilon A$. Since each α_t is an A-matrix, we conclude from proposition 2.4. that $\alpha_1\alpha_2 \cdot \cdot \alpha_t$ can be expressed as a T_A-polymomial $\Psi(\alpha_1, \alpha_2 \cdots \alpha_t)$ of degree less than t. Note that each factor in $\alpha_1\alpha_2\cdots\alpha_t=\Psi(\alpha_1, \alpha_2, \alpha_t)$ belongs to E, since $\alpha_i \varepsilon E$ and $T_A \subset E$. Therfore, we apply the homomorphism $\hat{\alpha}_i$ to this expression, and letting the resulting homomorphism of S act. on $1 \varepsilon S$. We see that

$$1\hat{\alpha}_1\hat{\alpha}_2\cdots\hat{\alpha}_t=s_1 s_2 \cdots s_t$$

Hence

$1 \hat{d}_1 \hat{d}_{,1} \hat{d}_2 \cdots \hat{d}_{ir} \hat{d}_{r+1} = a_1 s_{i1} a_2 \cdots s_{ir} a_{r+1}$ where $0 \leqq r < t$ and $a_j = d_j \varepsilon A$ and at least one $a_j \varepsilon A$ occurs. In other words $s_1 s_2 \cdots s_t$ can be expressed as an A-polynomial in $s_1, s_2, \cdot s_t$ of degree less than t. The theorem is proved.

Ⅲ. Jacobson Radical

In this paper the Jacobson Radical of a ring R is denoted by $J(R)$ and means that $J(R)$ is the set of all elements of R which annihilate all the irreducible R-modules. At first we know the following theorem from theorm 2.5.

Theorem 3.1. *Let* $R \subset S = \sum_{i=1}^{n} Rx_i$, $x_i R = Rx_i$ *be a finite normaling extension of rings. If A is a proper right ideal of R, then AS is a proper right ideal of S.*

Proof. Clearly AS is a ideal of S. Assume that $1 \varepsilon AS$. By theorem 2.5. $1=1^t=\Psi(1,1,\cdots 1)$, thus 1 is a sum of monomials in suitable elements of A. Consequently $1 \varepsilon A$.

Corollary 3.2. *If S is a finite normalizing exension of R and S is a division ring, then R is division ring.*

Proof. If R has an proper right ideal A, then AS is a proper right ideal of S. This is contradiction. Hence R is a division ring.

Corollary 3.3. *Any irreducible right R-module V can be embedded in a suitable irreducible right S-module W.*

Proof. We can write $V=R/A$ with A a maximal right ideal of R. Choose a maximal right ideal B of S with $AS \subset B$. This is possible by theorem 3.1. Then $W=S/B$ is irreducible and contains V since $B \cap R = A$.

Now we will get the main thoerm that is $J(R)=J(S) \cap R$.

Theorem 3.4. *If S is a finite normalizing extension of R then* $J(R)=J(S)\cap R$.

Proof. Let $r \varepsilon J(S) \cap R$ and let V be an irreducible right R-module. Then, as we have observed above we can find an irreducible right S-module W with $V \subset W_R$. Since $Wr=0$, it follows that $Vr=0$. Thus r annihilate each irreducible right R-module and hence belongs to $J(R)$.

Conversesly, let $r \varepsilon J(R)$ and let W be an irreducible right S-module, then by a result of Formanek and Jategaonkar, the restricted module W_R is completely reducible [3]. Since r annihilates the irreducible components of W_R, we conclude that $Wr=0$. Thus $r \varepsilon J(S)$, and the thorem is proved.

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