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Functions with closed Graph and some other related Properties

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〈Abstract〉

P. E. Long introduced a function with closed graph. We will investigate the function with closed graph by means of nearly open mappings due to V. P. Singh and study its characterizations.

폐 그래프를 갖는 함수와 그 성질에 관하여

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〈요　약〉

P. E. Long은 폐그래프를 갖는 함수를 도입했었다. 우리는 그 함수를 근사 개 함수를 이용하여 그 성질을 연구한다.

1. In a recent paper [7] the authors have studied in detail the concept of nearly open mappings. During this study, it was found that such mappings which in addition have closed graph present some interesting features. For instance, it was found that if $f: X \to Y$ is a surjective nearly open mapping with closed graph, then Y must be a T_2-spaces. Such results had been investigated earlier in the paper of Long [4]. In these papers they have investigated several interesting conditions on the mapping so as to imply that either X or Y or both are T_1/T_2-spaces. On the other hand R. V. Fuller in [6] has drawn attention to the mapping having closed point inverses, a concept weaker than the mapping having closed graph, and has investigated their properties.

In this paper, we have investigated, on one hand the implications of mappings which in addition to having some other properties have either closed point inverses or compact point inverses or are such that inverses of closed sets are compact. In this context, we have also investigated the role of nearly open and nearly continuous mappings. On the other hand we have investigated some conditions under which a given mapping has the property that its graph is closed.

2. Let X, Y be topological spaces and let $f: X \to Y$ be a mapping. We give below some definition which will be useful in the context of this papers.

(i) The subset $G_f = \{(x, f(x)) : x \in X\}$ of the product $X \times Y$ is called the graph of f. The graph of f is closed if for each $x \in X$, $y \neq f(x)$ there exist open sets U and V containing x and y respectively such that $f(U) \cap V = \emptyset$ [4]. If U and V are such that $f(U) \cap \overline{V} = \emptyset$, then the

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graph of f is said to be strongly closed[5].

(ii) f is said to be nearly open if for every open set U in X, $f(U)\subset \mathrm{Int}(\overline{f(U)})$. f is called nearly continuous if for every open set $V\subset Y$, $f^{-1}(V)\subset \mathrm{Int}(\overline{f^{-1}(V)})$.

(iii) f is said to have closed point inverse if $f^{-1}(y)$ is closed in X for each $y\in Y$. Similarly, f is said to have compact point inverses if $f^{-1}(y)$ is compact in X for each $y\in Y$.

(iv) A topological space X is called a KC-space if every compact subset of X is closed [1].

The remaining terminology used is standard and is drawn mainly from [3].

3. We begin with the study of some implications of the type referred to in [1] in case of certain mappings with closed point inverses.

Proposition 3.1: Let $f: X\to Y$ be surjective closed or open mapping having closed point inverses. Then for any $A\subset Y$, $y\notin A$ implies that $A\cap\{\bar{y}\}=\emptyset$.

Proof: Case (1). Let the surjective mapping $f: X\to Y$ with closed point inverses be closed. Clearly for $y\notin A$, $f^{-1}(y)\cap f^{-1}(A)=\emptyset$, $f^{-1}(A)\subset X-f^{-1}(y)$. $f^{-1}(y)$ is closed and f is a closed mapping, so from [3 page 86] there exist an open set U containing A such that

$f^{-1}(A)\subset f^{-1}(U)\subset X-f^{-1}(y)$

or $f^{-1}(U)\cap f^{-1}(y)=\emptyset$

or $y\notin U\supset A$, i.e. $A\cap\{\overline{Y}\}=\emptyset$

Case 2. Let $f: X\to Y$ be the surjective open mapping having closed point inverses. Then for $y\notin A$, $f^{-1}(A)\subset X-f^{-1}(y)$, $X-f^{-1}(y)$ is open so there exists open set U such that $f^{-1}(A)\subset U\subset X-f^{-1}(y)$ or $U\cap f^{-1}(y)=\emptyset$, $y\notin f(U)\supset A$. f is open so $f(U)$ is open neighbourhood of A not containing y such that $A\cap\{\bar{y}\}=\emptyset$.

The following simple and evident characterization (most probably known) of T_1 spaces is useful in this context.

Proposition 3.2: Any topological space Y is T_1 if for any set $A\subset Y$, and $y\notin A$ there exist an open set U containing A such that $y\notin U$.

The propositions 3.1 and 3.2 immediately give the following corollary:

Corollary 3.3: $f: X\to Y$ is a surjective closed or open mapping having closed point inverses, then Y is a T_1-space.

Proposition 3.4: If $f: X\to Y$ is an injective map with closed point inverses, then X is T_1.

Proof follows immediately from the fact that for each $x\in X$, $x=f^{-1}(f(x))$ and so $\{x\}$ is closed in X.

Combining corollary 3.3 and proposition 3.4 we are led to the following proposition which improves to some extent the theorem 5 in [4] of Long.

Proposition 3.5: If $f: X\to Y$ is a bijective closed or open map with closed point inverses, then both X and Y are T_1-spaces.

Long has shown that if $f: X\to Y$ is a surjective map with closed graph, then Y is a T_1-space and if f is an open surjective map with closed graph, then Y is T_2-space. We see that if instead we take the mapping to be a closed surjective mapping with closed graph, then Y turns out to be KC-space of Willanski defined above in [2].

Proposition 3.6: Let $f: X\to Y$ be a closed surjective map with closed graph. Then Y is a KC-space.

Proof: It is enough to prove that for any compact set $F\subset Y$, $y\notin F$ we have $y\notin\overline{F}$. For $y\notin F$ we have $f^{-1}(y)\cap f^{-1}(F)=\emptyset$ or $f^{-}(y)\subset X-f^{-1}(F)$. Graph of f is closed and so from theorem 3.6 in [6], $f^{-1}(F)$ is closed in X. As the given mapping is closed we get an open set U containing y such that $f^{-1}(y)\subset f^{-1}(U)\subset X-f^{-1}(F)$ or $f^{-1}(U)\cap f^{-}(F)=\emptyset$ or $U\cap F=\emptyset$ which implies that $y\notin\overline{F}$. Hence Y is a KC-space.

4. In this section we investigate the role of nearly open and nearly continuous mappings in the context of problems of the type-investigated in this paper. As already mentioned above, it

is known that the range of a surjective open mapping with closed graph is T_2-space [4]. We first see that this result holds if open mappings are replaced by nearly open mappings.

Proposition 4.1: Let $f: X \to Y$ be nearly open surjection with closed graph, Then Y is a T_2-space.

Proof: Let $y, y_1 \in Y$ such that $y \neq y_1$. f is surjective so there exists $x_1 \in X$ such that $y_1 = f(x_1)$. Since the graph of f is closed, for $y \neq f(x_1)$ there exist open neighbourhood U of x_1 and V of y such that $f(U) \cap V = \emptyset$. As f is nearly open we have

$y_1 \in f(U) \subset \mathrm{Int}(\overline{(f(U)}) \subset \overline{f(U)} \subset Y - V$.

Thus $\mathrm{Int}(\overline{f(U)})$ & V are the required disjoint neighbourhoods of y_1, & y respectively. Hence Y is T_2-space.

Remark 4.1.1: Since $\mathrm{Int}(\overline{f(U)}) \cap V = \emptyset$ in the above result. It follows that $f(U) \cap \overline{V} = \emptyset$ and so the hypothesis of above proposition implies that f has a strongly closed graph in the sense of Long and Herrington [5].

We can similarly prove the following proposition which involves the condition under which X becomes T_2.

Proposition 4.2: If $f: X \to Y$ is an injective and nearly continuous mapping with closed graph, then X is T_2.

Proof: Let $x_1 \neq x_2$ then $f(x_1) \neq f(x_2) = y$(say).

From closed graph condition there exist open sets U containing x_1, and V containing y such that $f(U) \cap V = \emptyset$

or $U \cap f^{-1}(V) = \emptyset$

or $U \cap \overline{f^{-1}(V)} = \emptyset$, f is nearly continuous, it follows that $U \cap \mathrm{Int}((\overline{f^{-1}(V)}) = \emptyset$ and these are disjoint open neighbourhoods of x_1 and x_2 respectively. Hence X is T_2.

Combining propositions 4.1 and 4.2 we are led to the following proposition which is slight improvement of the theorem 8 in [4].

Proposition 4.3: Let $f: X \to Y$ be a bijective nearly continuous and nearly open mapping having closed graph. Then both X and Y are T_2 spaces.

5. In this context, it is natural to pose the question as to whether by choosing a suitable mapping $f: X \to Y$ the range space Y becomes regular or T_3. We find that the mapping under which inverses of closed set are compact, provide the answer to this question. For the sake of convenience we shall call such mappings, mappings having inverses of closed sets compact. Such mappings must be well known, though we have not been able to trace any reference to them in the literature accessible to us. It can be easily seen that these mappings are distinct from continuous mappings whenever the domain is not a KC space. We now prove the following proposition:

Proposition 5.1: Let $f: X \to Y$ be nearly open surjection with closed graph and having inverses of closed sets compact. Then Y is T_3.

Proof: Let F be any closed set in Y and $y \notin F$, then $\{f^{-1}(F) \times \{y\}\} \cap G_f = \emptyset$. For if not so, then we have $z \in f^{-1}(F) = M$(say) such that $y = f(z) \in F$ which is a contradiction. From the closed graph condition for each $(m_i, y) \in M \times \{y\}$ there exist open sets U_i and V_i containing m_i, y respectively such that

$(U_i \times V_i) \cap G_f = \emptyset$. M is compact so open cover $\{U_i\}$ has finite sub-cover $U_1, U_2 \cdots U_n$. Setting $U = \bigcup_{i=1}^{n} \{U_i\} \supset M$ and $V = \bigcap_{i=1}^{n} \{V_i\} \ni y$, we have

$f(U) \cap V = \emptyset$ or $\overline{f(U)} \cap V = \emptyset$. As f is nearly open, we have $\mathrm{Int}(\overline{f(U)}) \cap V = \emptyset$.

Thus $\mathrm{Int}(\overline{f(U)})$ & V are the required disjoint neighbourhoods of F and y respectively. Since the hypothesis of the proposition imply that Y is T_2 also, it follows that Y is T_3.

Corollary 5.2. Let $f: X \to Y$ be a nearly open surjection with closed graph and having inverses of closed sets compact.

Then each pair of disjoint subsets A and B of Y, one of which is closed and other compact, can always be separated by disjoint neighbourhoods.

6. Fuller [6] has shown that if X is regular

and $f: X \to Y$ is a closed mapping with closed point inverses then f has a closed graph, we find that if instead we assume X to be T_2 we have to take the mapping with compact inverses instead to get the same conclusion.

Proposition 6.1: If X is T_2 and the mapping $f: X \to Y$ is a closed mapping with compact-point inverses, then f has closed graph.

Proof: It is sufficient to show that for $y \neq f(x)$ there exist open sets U and V containing x and y respectively such that $f(U) \cap V \neq \emptyset$ Now for $y \neq f(x)$, $x \not\in f^{-1}(y)$ and $f^{-1}(y)$ is compact. Since X is T_2 there exist open sets U containing X and U' containing $f^{-1}(y)$ such that $U \cap U' = \emptyset$. f is closed, therefore corresponding to $f^{-1}(y) \subset U'$ there exists an open set V containing y such that $f^{-1}(y) \subset f^{-1}(V) \subset U'$. We have therefore $U \cap f^{-1}(V) = \emptyset$ or $f(U) \cap V = \emptyset$ and this implies that the graph of f is closed.

Corollary 6.2: Any bijective closed mapping $f: X \to Y$ has closed graph whenever X is T_2.

7. In a recent discussion with Professor T. Sounderajan of Madurai University, Madurai, India with reference to our work published elsewhere [8] two questions emerged. Answers to these questions and some related results are given below.

Question 7.1; Let $f: X \to Y$ have closed graph, Let $a \in X$, $b = f(a)$. Is $b = \cap \{\overline{f(U)}: U$ is a neighbourhood of $a\}$.

Question 7.2: Let $f: X \to Y$ be the mapping such that $\forall\ a \in X$, $f(a) = \cap \{\overline{f(U)}: U$ is a neighbourhood of $a\}$. Should G_f be closed?

Answers to Question 7.1: Let $z \in \cap \{\overline{f(U)}: U$ is a neighbourhood of $a\}$ and $z \neq b$. This implies that $z \in \overline{f(U)}$ for every neighbourhood U of a or for every neighbourhood V of z and U of a, $V \cap f(U) \neq \emptyset$. This implies that $(U \times V) \cap G_f \neq \emptyset$ or $(a, z) \in \overline{G}_f = G_f$ or $f(a) = z$, and hence $b = \cap \{\overline{f(U)}: U$ is a nbd of $a\}$,

Answer to Question 7.2: To prove that graph of f, G_f, is closed, it suffices to show that whenever $(x, y) \not\in G_f$ we have $(x, y) \not\in \overline{G}_f$. Let $(x, y) \not\in G_f$ and put $y \neq f(x) = b$(say). From the given condition $y \neq b = \cap \{\overline{f(U)}: U$ is a neighbourhood of $x\}$ we have $y \not\in \overline{f(U)}$ for at least one nbd U of x. This implies that there exists a nbd V of y such that $V \cap f(U) = \emptyset$ or $(x, y) \not\in \overline{G}_f$. Hence $G_f = \overline{G}_f$ or the graph of f is closed.

Remark 1: Answer to the question 7.1 gives immediately a condition on $f: X \to Y$ in order that Y becomes T_1. If $f: X \to Y$ is a surjective mapping with closed graph then the conclusion of question 7.1 implies that each point of Y is closed and hence Y is T_1. The following proposition can consequently be stated:

Let $f: X \to Y$ be a surjective map with closed graph. Then Y is T_1.

Remark 2: The answers to question 7.1 and 7.2 give a necessary and sufficient condition in order that graph of a mapping, $f: X \to Y$ be closed. This can be summarized as follows.

A mapping $f: X \to Y$ has closed graph iff for each $a \in X$, $f(a) = \cap \{\overline{f(U)}: U$ is a nbd of $a\}$.

References

[1] A. WILLANSKI—Between T_1 and T_2. Amer. Math. Monthly (1967) pp. 261—266.

[2] B.J. PETTIS—Some Topological Questions related to open mapping and closed Graph Theorem. Studies in Topology, Academic Press, New York(1975). pp. 451—456.

[3] JANES DUGGNDJI—Topology, Allyn and Becon Series.

[4] P.E. LONG—Functions with closed graphs. Amer. Math. Monthly 76(1969) pp. 930—932.

[5] P.E. LONG AND L.L. HERRINGTON—Functions with Strongly closed Graphs. Bolletino U. MI(4) (1975) pp. 381—384.

[6] R.V. FULLER—Relations among continuous and various non-continuous functions, Pacific Journal of Math. Vol. 25 (1968) pp. 495—509.

[7] V.P. SINGH AND D.N. MISRA—Nearly Open Mappings and a Problem of Franklin. (Under communication).

[8] V.P. SINGH AND D.N. MISRA—A note on Topological Group (To appear in Math. Seminar Notes Rokko, Kobe, Japan).