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A Note on a Maximum Theorem for Harmonic Functions

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〈Abstract〉

Let $u(r,\theta)$ be harmonic in $D=\{|z|<1\}$ and let $\lim_{r\to 1}\inf u(r,\theta)$ be non-negative for almost all θ in $0\leqq\theta\leqq 2\pi$. Then, unless $-\infty$ is anasymptotic value of $u(r,\theta)$ for at least one point $e^{i\theta}$ on $|z|=1$, $u(r,\theta)$ is non-negative in D.

This Result is known; the proof, however, given by Lohwater, Bruckner, and Ryan (1), uses deep results from the theory of real functions; a direct proof by complex analysis is given.

조화함수의 최대치 정리에 관한 노우트

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〈요 약〉

Lohwater, Bruckner, Ryan (1) 이 발표한 단위원내에서 조화인 함수의 최대치 정리와 관련된 결과를 복소수함수론의 cluster set을 써서 직접 간결하게 증명했다.

The above result is a direct consequence of the following simple theorem;

Let $u(r,\theta)$ be harmonic in D and let $\lim_{r\to 1}\sup u(r,\theta)$ be bounded above by m for almost all θ in $0\leqq\theta\leqq 2\pi$. Then, unless ∞ is an asymptotic value of $u(r,\theta)$ for at least one point $e^{i\theta}$ on $|z|=1$, $u(r,\theta)$ is bounded above by m.

Proof. Let $\Gamma=\{z: |z|=1\}$ and let E be the set of points of Γ at which we do not known whether $\lim_{r\to 1}\sup u(r,\theta)$ is bounded above by m. Let $v(r,\theta)$ be the harmonic conjugate of $u(r,\theta)$. Then the function $f(r,\theta)=e^{u(r,\theta)+iv(r,\theta)}$ is an analytic function in D. We show that thisf unction is bounded in D. Assume the contrary. Then there exists a point z_0 on Γ such that $\infty\in C_D(f,z_0)$ where $C_D(f,z_0)$ denotes the cluster set of $f(r,\theta)$ at the point z_0. Consider Noshiro's boundary cluster set[2]

$$C^*_{\Gamma-(E\cup\{z_0\})}(f,z_0)$$

formed by means of theradii on which $\lim_{r\to 1}\sup u(r,\theta)$ is bounded above by m. Since $\lim_{r\to 1}\sup u(r,\theta)\leq m$ for every point in $\Gamma-E$, it follows that ∞ does not belong to this cluster set. And since $f(r,\theta)$ is analytic in D, it doe snot assume ∞ in D. Hence, by Noshiro's theorem[3] ∞ is an asymptotic value of $f(r,\theta)$ at some point of Γ, which is contrary to hypothesis.

Thus $f(r,\theta)$ is bounded in D and by Fatou's theorem, $f(r,\theta)$ has a radial limit at almost every point of Γ. At almost every such point the modulus of the radial limit is less than or equal to e^m. Let E_m be the set of all such points.

Then $f(r,\theta)$ can be repressented as

$$f(r,\theta)=\frac{1}{2\pi}\int_{E_m} f(e^{it})\ P(\theta-t:r)\ dt.$$

Thus

$$|f(r,\theta)| \leq \frac{1}{2\pi}\int_{C_m} |f(e^{it})| P(\theta - t : r)\ dt$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} P(\theta - t : r)\ dt \leq e^m,$$

that is,

$e^{u(r,\theta)} \leq e^m$ or $u(r,\theta) \leq m$.

References

1. LOHWATER, A. J., Bruckner, A. M. & Frank Ryan, Some Non-negativity Theorems for Harmonic Functions, pp. 3–8, Ann. Acad. Sci. Fenn. Helsinki (1969)

2. NOSHIRO, K., Cluster Sets, Springer-Verlag, Berlin (1960)

3. Ibid., p. 43.