울산대학교 자연과학연구논문집 제11권 2호 pp. 11~17, 울산대학교.
Journal of Natural Science Vol. 11, No. 2, pp. 11~17, 2002
University of Ulsan.

Properties of a certain reduced senigroup C^*-algebraa

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September 10, 2001

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⟨*Abstract*⟩

We show that the reduced semigroup C^*-algebra generated by the left regular representation of $\mathcal{S} = \{0, 2, 3, \dots\}$ acts irreducibly on $l^2(M)$, prime and its commutator ideal is the compact operator subalgebra of $\mathcal{B}(l^2(\mathcal{S}))$.

반군 대수의 성질

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⟨요약⟩

반군 $\{0, 2, 3, \dots\}$로 생성되는 반군 C^*-대수가 힐버트 공간 $l^2(\mathcal{S})$위에서 비 축퇴적으로 작용하며 프라임을 보였다.

1. Introduction

Let M denote a semigroup with unit e and $\mathcal{B}$ be a unital C^*-algebra. A map $W : M \to \mathcal{B}, x \mapsto W_x$ is called an *isometric homomorphism* if $W_e = 1$, W_x is an isometry and $W_{xy} = W_x W_y$ for all $x, y \in M$. If $\mathcal{B}$ is the $*$-algebra $\mathcal{B}(H)$ of all bounded linear operators of a non-zero Hilbert space H, we call (H, W) an *isometric representation* of M.

If M is left-cancellative, we can have a specific isometric representation of M as follows: let $l^2(M, H)$ denote the Hilbert space of all norm square-summable maps f from M to H. For each $x \in M$ we define a isometry $\mathcal{L}_x$ on $l^2(M, H)$ by the equation

$$(\mathcal{L}_x f)(z) = \begin{cases} f(y), & \text{if } z = xy \text{ for some } y \in M, \\ 0, & \text{if } z \notin xM, \end{cases}$$

for each $f \in l^2(M, H)$. The map $\mathcal{L} : M \to \mathcal{B}(l^2(M, H)), x \mapsto \mathcal{L}_x$ is clearly an isometric representation and we call it the left regular isometric representation of M. If we define an element $\widetilde{\xi}^{(x)} \in l^2(M, H)$ by setting

$$\widetilde{\xi}^{(x)}(y) = \begin{cases} \xi, & \text{if } y = x, \\ 0, & \text{otherwise}, \end{cases}$$

for $\xi \in H$ and $x, y \in M$, then $\mathcal{L}_y(\widetilde{\xi}^{(x)}) = \widetilde{\xi}^{(yx)}$ for all $x,\ y \in M$.

In order to make things explicit let us consider the semigroup $\mathbb{N}$ of all natural numbers, then $\mathcal{L}_1$ is the unilateral shift of $l^2(\mathbb{N})$. As we can see in the above statement, the left regular isometry is the linear operator to translate the orthonormal basis of $l^2(M)$, which has made it important for decades.

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[1]Mathematics Subject Classification: 46L05, 47C15, 47B35

[2]Key words and phrases: isometric homomorphism, left regular isometric representation, reduced semigroup C^*-algebra, semigroup C^*-algebra, Toeplitz algebra.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The C^*-algebras generated by isometries have been studied by many authors, ever since L. A. Coburn proved his well-known theorem, which asserts that the C^*-algebra generated by a non-unitary isometry on a separable infinite dimensional Hilbert space does not depend on the particular choice of the isometry [1]. In particular, many authors have interest in the generalization of Coburn's theorem called the uniqueness property of the C^*-algebras generated by isometries. If the C^*-algebras generated by isometries have the uniqueness property, the structures of those C^*-algebras are to some extent independent of the choice of isometries on a Hilbert space. All the C^*-algebras generated by the isometric representations of the semigroup $\mathbb{N}$ of all natural numbers have the uniqueness property and so are isomorphic to the Toeplitz algebra by Coburn's result. In addtion, it was known that the uniqueness property holds for the C^*-algebras generated by one-parameter semigroups of isometries [5] and the Cuntz algebras [2], but there are few known C^*-algebras with the uniqueness property except these C^*-algebras. And so for the lack of examples of the C^*-algebras with the uniqueness property, the uniqueness property was modified in several ways [13] and we also have interest in the uniqueness property which is modified in this paper.

One of the ways to construct the C^*-algebras generated by isometries is to consider the isometric representations of the semigroups and C^*-algebras generated by them. Among these the C^*-algebras generated by the left regular isometric representations of the left-cancellative semigroups have been studied much for decades [1, 2, 3, 4, 5, 9, 11, 13,14, 16]. We are going to call it the reduced semigroup C^*-algebra from the point of the crossed products of C^*-algebras by semigroups of automorphisms and denote it $C^*_{red}(M)$ in this paper [8]. As a typical model of the reduced semigroup C^*-algebra we have the Toeplitz algebra $C^*_{red}(\mathbb{N})$ when the semigroup M is the semigroup $\mathbb{N}$ of all natural numbers. Besides the reduced semigroup C^*-algebra we can consider the semigroup C^*-algebra introduced by G. J. Murphy [12] which is obtained enveloping all isometric representations of M. Murphy denoted it by $C^*(M)$ and we also intend to use it. Seeing from the definition of the semigroup C^*-algebra, the semigroup C^*-algebra has the universal property as follows: if we put the canonical isometric homomorphism V of M to the semigroup C^*-algebra $C^*(M)$, then for any isometric homomorphism W of M to a unital C^*-algebra B there exists a unique homomorphism from $C^*(M)$ to the unital C^*-algebra B sending V_x to W_x for each $x \in M$.

Actually, if the reduced semigroup C^*-algebra $C^*_{red}(M)$ and the semigroup C^*-algebra $C^*(M)$ are isomorphic, the left regular isometric representation of M has the universal property of the isometric representations of M. Many authors have interests in when these two C^*-algebras $C^*_{red}(M)$ and $C^*(M)$ are isomorphic or when $C^*_{red}(M)$ has the universal property of some kinds of isometric representations of M, which are examples of the modified uniqueness property

In this paper we show that the problem when $C^*_{red}(M)$ and $C^*(M)$ are isomorphic is much dependent on the order structure of M by analyzing the structure of $C^*_{red}(\mathcal{S})$ and $C^*(\mathcal{S})$ where $\mathcal{S} = \{0, 2, 3, \dots\}$.

The semigroup $\mathcal{S} = \{0, 2, 3, \dots\}$ is the generating subsemigroup of the integer group $\mathbb{Z}$ and the semigroup $\mathbb{N} = \{0, 1, 2, \dots\}$ is same. But the order structure of $(\mathbb{Z}, \mathcal{S})$ with the positive cone $\mathcal{S}$ is different from that of $(\mathbb{Z}, N)$. Though it is known that $C^*_{red}(\mathbb{N})$ is isomorphic to $C^*(\mathbb{N})$ by Coburn's result, we show that $C^*_{red}(\mathcal{S})$ is not isomorphic to $C^*(\mathcal{S})$ by using the order structure of $\mathcal{S}$ in Proposotion 2.7. Furthermore we can say that if the order of the semigroup M is not unperforated, $C^*_{red}(M)$ is not isomorphic to $C^*(M)$ from the structure of $C^*_{red}(\mathcal{S})$ and $C^*(\mathcal{S})$ in Theorem 2.6.

2. Reduced semigroup C^*-algebra $C^*_{red}(\mathcal{S})$

We can give an order on M as follows: if an element x in M is contained in yM for some element $y \in M$, then x and y are comparable and we denote it by $y \leq x$. This relation makes M a pre-order semigroup. If the unit of M is the only invertible element of M, the above relation on M becomes a partial order on M. And we can say a maximal and a minimal element in M in the following sense ; an element $a_0 \in M$ is maximal if and only if $a_0 \leq x$ implies $x = a_0$ and an element a_1 is minimal if and only if $x \leq a_1$ implies $a_1 = x$ for $x \in M$.

Let $\mathcal{S} = \{0, 2, 3, \ldots\}$, then the ordered group $(\mathbb{Z}, \mathcal{S})$ is a partially ordered group and not unperforated, so the order structure of $(\mathbb{Z}, \mathcal{S})$ is different from that of $(\mathbb{Z}, \mathbb{N})$.

Since $C^*_{red}(\mathcal{S})$ is the closed linear span of $\{\mathcal{L}_{n_1}\mathcal{L}^*_{n_2} \cdots \mathcal{L}_{n_{2k-1}}\mathcal{L}^*_{n_{2k}} \mid n_j \in \mathcal{S}\}$, we look at how the left regular isometry $\mathcal{L}_n$ acts on $l^2(\mathcal{S})$ for each $n \in \mathcal{S}$.

If we define a map δ_n by the equation for each $n \in \mathcal{S}$,

$$\delta_n(m) = \begin{cases} 1, & m = n, \\ 0, & \text{otherwise,} \end{cases}$$

then $\{\delta_n \mid n \in \mathcal{S}\}$ is the canonical orthonormal basis of $l^2(\mathcal{S})$. And we have $\mathcal{L}_n(\delta_m) = \delta_{n+m}$ for $n, m \in \mathcal{S}$.

We put $P_n = \mathcal{L}_n\mathcal{L}^*_n$ and $Q_n = I - P_n$ for each $n \in P$

PROPOSITION 2.1 The projection P_n is the orthogonal projection onto the closed linear span of $\{\delta_n, \delta_{n+2}, \ldots\}$ and Q_n is the orthogonal projection onto the closed linear span of $\{\delta_0, \delta_2, \delta_3, \ldots, \delta_{n-1}\}$.

PROOF If $m \leq n$ for each $m, n \in \mathcal{S}$, then

$$P_n(\delta_m) = \mathcal{L}_n\mathcal{L}^*_n(\delta_m) = \mathcal{L}_n(\delta_{m-n} = \delta_m.$$

Since $m \leq n$ implies that $m - n \in \mathcal{S}$, $m \leq n$ if and only if $m \in \{n, n+2, n+3, \ldots\}$. if m is not comparable with n or $m \geq n$, then $P_n(\delta_m) = 0$. Therefore P_n is the orthogonal projection onto the closed linear span of $\{\delta_n, \delta_{n+2}, \delta_{n+3}, \cdots\}$ and $O_n = I - P_n$ is the orthogonal projection onto the closed linear span of $\{\delta_0, \delta_2, \ldots, \delta_{n-1}\}$. □

Let $\mathcal{B}$ be the C^*-subalgebra of $C^*_{red}(S)$ generated by P_n for all $n \in P$ and $\mathcal{Z}(C^*_{red}(S))$ the ideal of $C^*_{red}(S)$ generated by Q_n for all $n \in P$.

The group C^*-algebra of an abelian group is, of course, it self abelian and so not very interesting from the point of view of C^*-theory. But the reduced semigroup C^*-algebras and the semigroup C^*-algebras may not be abelian and moreover primitive for a large abelian class of semigroups.

We can see there exists no non-trivial reducing subspace of $l^2(P)$ for $C^*_{red}(S)$ by the following proposition.

PROPOSITION 2.2. $C^*_{red}(P)$ acts irreducibly on $l^2(\mathcal{S})$.

PROOF. Assume that the operator T in $\mathcal{B}(l^2(\mathcal{S}))$ commutes with $C^*_{red}(\mathcal{S})$. Let $[T_{n,m}]$ denote the matricial representative with respect to the canonical orthonormal basis $\{\delta_n\}$ of $l^2(\mathcal{S})$. Then

$$\begin{aligned} T_{n,m} &=< T(\delta_m), \delta_n > \\ &=< T(\delta_m), \mathcal{L}_n\delta_0 > \\ &=< \mathcal{L}_n^* T(\delta_m), \delta_0 > \\ &=< T\mathcal{L}_n^*(\delta_m), \delta_0 > . \end{aligned}$$

Similarly $T_{n,m} =< T\mathcal{L}_m\delta_0, \delta_n >=< T\delta_0, \mathcal{L}_m^*\delta_n >$. Hence $T_{n,m} = 0$ if n is not equal to m, so T is a diagonal operator. Furthermore we can have that $T_{n,n} = T_{0,0}$ for all $n \in \mathcal{S}$ from the following equation

$$T_{n,n} =< T\mathcal{L}_n(\delta_0), \mathcal{L}_n(\delta_0) >=< \mathcal{L}_n^*\mathcal{L}_n T(\delta_0), \delta_0 >=< T(\delta_0), \delta_0 > .$$

It follows that $C^*_{red}(S)$ acts irreducibly on $l^2(\mathcal{S})$.

□

PROPOSITION 2.3. The commutator ideal $\mathcal{Z}(C^*_{red}(\mathcal{S}))$ of $C^*_{red}(\mathcal{S})$ is the compact operator algebra $\mathcal{K}(l^2(\mathcal{S}))$.

PROOF. Since $C^*_{red}(\mathcal{S})$ is generated by $\mathcal{L}_2$ and $\mathcal{L}_3$, it is enough to see how these operators act on $l^2(\mathcal{S})$. The operator $I - \mathcal{L}_2\mathcal{L}_2^*$ is of finite rank, so contained in $\mathcal{K}(l^2(\mathcal{S}))$. Therefore $\mathcal{K}(l^2(\mathcal{S}))$ and the commutator ideal $\mathcal{Z}(C^*_{red}(\mathcal{S}))$ have non-empty intersection. Since $C^*_{red}(S)$ acts irreducibly on $l^2(\mathcal{S})$ by the Proposition 2.2, $\mathcal{Z}(C^*_{red}(\mathcal{S}))$ acts also irreducibly on $l^2(\mathcal{S})$. Therefore the commutator ideal $\mathcal{Z}(C^*_{red}(\mathcal{S}))$ contains the compact operator algebra $\mathcal{K}(l^2(\mathcal{S}))$ because $\mathcal{Z}(C^*_{red}(\mathcal{S}))$ and $\mathcal{K}(l^2(\mathcal{S}))$ have non-empty intersection [15].

Furthermore $C^*_{red}(\mathcal{S})/\mathcal{K}(l^2(\mathcal{S}))$ is abelian because $I - \mathcal{L}_2\mathcal{L}_2^*$ and $I - \mathcal{L}_3\mathcal{L}_3^*$ are contained in $\mathcal{K}(l^2(\mathcal{S}))$. Hence $\mathcal{Z}(C^*_{red}(\mathcal{S}))$ is equal to $\mathcal{K}(l^2(\mathcal{S}))$.

A C^*-algebra $\mathcal{A}$ is simple if $\mathcal{A}$ has no non-trivial closed ideal of $\mathcal{A}$ and prime if any two non-zero closed ideals of $\mathcal{A}$ have non-zero intersection. The prime C^*-algebras and the simple C^*-algebras play an important role in the theory of the structure of the C^*-algebras because the prime C^*-algebras and the simple C^*-algebras are the analogs of factors in the theory of von Neumann algebras.

Though there are many interesting simple group C^*-algebras, the reduced semigroup C^*-algebras are rarely simple for a large and natural class of semigroups. The facts which we have interest in are that there are abundantly prime reduced semigroup C^*-algebras and that it is still open when the reduced semigroup C^*-algebra is prime.

PROPOSITION 2.4. $C^*_{red}(\mathcal{S})$ is prime

PROOF. Let J be a non-zero ideal of $C^*_{red}(S)$. If x is a non-zero element in J, xk is a compact operator for each $k \in \mathcal{K}(l^2(\mathcal{S}))$. Since J is also irreducible because of the irreducibility of $C^*_{red}(S)$, $\mathcal{K}(l^2(P))$ is contained in J. Therefore, if I and J are non-zero ideals in $C^*_{red}(S)$, then I and J have a non-zero intersection. So $C^*_{red}(S)$ is prime. □

PROPOSITION 2.5. $C^*_{red}(S)$ is primitive.

PROOF. Since $C^*_{red}(S)$ acts irreducibly on $l^2(\mathcal{S})$, we can see the identity map from $C^*_{red}(S)$ to $\mathcal{B}(l^2(\mathcal{S}))$ as a faithful irreducible representation of $C^*_{red}(S)$. □

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