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Measurability on the Ordered Topological Vector Lattice.

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〈Abstract〉

The basic properties of the I-summable class $S(I)$ on an ordered topological vector lattice has been studied by H. Anton and W. J. Pervin. Extending their methods in this paper, we are going to give some fundamental theorems which are the analogues for the positive functional I of Fatou 's lemma and Lebesgue convergence theorem.

Finally, we introduce I-measurable on the ordered topological vector lattice.

순서 위상 벡터 격자위에서 정의된 가측도

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〈요 약〉

순서 위상 벡터 격자위에서 정의된 가합의 기본성질은 H. Anton과 W. J. Pervin에 의하여 연구되었다. 이 논문에서는 그들의 방법을 확장하여 양범함수에 관한 Fatou 보조정리와 Lebesgue수렴정리에 유사한 몇개의 기본정리를 보이고, 마지막으로 순서위상벡터 격자위에서 정의된 가측의 성질을 소개하고자 한다.

I. Preliminaries and notations

We shall introduce terminologies and notations. Let E be a fixed ordered topological vector lattice with the real field throughout this paper. A nonempty subset L of E is called an ***integration lattice*** if L is a vector lattice and for every $a\in E$ there exists an element $\hat{a}\in L$ such that $a\leqq\hat{a}$. If L is an integration lattice, a strictly positive linear functional $I: L\rightarrow R$ is called an ***elementary integral*** if $I(a_n)\rightarrow 0$ whenever $\{a_n\}$ is a sequence in L such that $a_n\downarrow 0$ with respect to the topological vector space. For simplicity we will assume that L is a fixed integration lattice on an ordered topological vector lattice E, and that I is an elementary integral on L.

A point $a\in E$ will be called an ***upper element*** if there exists a sequence $\{a_n\}$ in L such that $a_n\uparrow a$. The class of all upper elements will be denoted by U. From the continuity of the linear and lattice operations, it is obvious that the class U is a vector lattice and that I is well-defined on U. We can extend I from L to U by defining $I(a)=\lim I(a_n)$, where $\{a_n\}$ is any sequence in L such that $a_n\uparrow a$.

It follows from the second condition of the integration lattices that I is finite valued. It is also easy to see that $I(a+b)=I(a)+I(b)$ and $I(ka)=kI(a)$ for $k\geqq 0$ and a, b in U. The ***positive cone*** of E will be denoted by P. Here we shall prepare the following theorem which has been proved by H. Anton and W. J. Pervin (5).

Theorem 1.1. (i) I is *monotone and strictly positive on* U. (ii) *If* $\{a_n\}$ *is a sequence in* U *such that* $a_n \uparrow a$, *then* $a \in U$ *and* $I(a_n) \to I(a)$.

Now define $-U$ by $-U = \{a \in E: -a \in U\}$. It is easy to show that $-U$ is a vector lattice and that $a \in -U$ if and only if there exists a sequence $\{a_n\}$ in L such that $a_n \downarrow a$. If $a \in -U$, we define $I(a)$ by $I(a) = -I(-a)$. This definition extends I as a monotone and strictly positive functional from U to $U \cup -U$ and for $a, b \in -U$ and $k \geqq 0$, we have $I(a+b) = I(a) + I(b)$ and $I(ka) = kI(a)$. Further, if $a_n \in -U$ and $a_n \downarrow a$, then $a \in -U$ and $I(a_n) \to I(a)$. H. Anton and W. J. Pervin also introduced the following definition in(5).

Definition 1.2. An element a in the topological vector lattice E is said to be *I-summable* if given any $\varepsilon > 0$ there exists a pair of elements $x \in -U$ and $y \in U$ such that $x \leqq a \leqq y$ with $I(y) - I(x) < \varepsilon$. The class of *I-summable elements* will be denoted by $S(I)$.

Definition 1.3. Let a be an element in E. The extended real numbers $\bar{I}(a) = \inf\{I(y): y \geqq a,\ y \in U\}$ and $\underline{I}(a) = \sup\{I(x): x \leqq a,\ x \in -U\}$ are called the *upper* and *lower elementary integrals* of a, respectively.

It is clear that if $a \in S(I)$ then $\bar{I}(a)$ and $\underline{I}(a)$ are equal. We define $I(a)$ to be this common (finite) value. We note that if $a \in U \cup -U$, then $a \in S(I)$ and the new definition of I agrees with the old. In particular, $L \subset S(I)$. The following remarkable fundamental theorem has been proved by H. Anton and W. J. Pervin (5)

Theorem 1.4. $S(I)$ *is an integration lattice and* I *is a strictly positive linear functional on* $S(I)$.

II. Fatou 'S lemma and Lebesgue convergence theorem in $S(I)$

Lemma 2.1. *An element* x *in* P *belong to* U *if and only if there is a sequence* $\{a_k\}$ *in* $P \cap L$ *such that* $x = \sum_{k=1}^{\infty} a_k$. *In this case* $I(x) = \sum_{k=1}^{\infty} I(a_k)$.

Proof. We set $\sum_{k=1}^{n} a_k = b_n$, then $b_n \in L$ for $n = 1, 2, \cdots$, since L is a linear subspace of E, and $b_n \uparrow x$. So the "if" part is trivial. On the other hand, let x be in P and $b_n \uparrow x$ with $b_n \in L$. Without loss of generality we may assume that each b_n is in P by replacing b_n by $b_n \vee 0$. Set $a_1 = b_1$, $a_n = b_n - b_{n-1}$ for $n > 1$, Then $x = \sum_{k=1}^{\infty} a_k$ and we have $I(x) = \lim I(b_n) = \lim I(\sum_{k=1}^{n} a_k)$ $= \lim \sum_{k=1}^{n} I(a_k) = \sum_{k=1}^{\infty} I(a_k)$.

Lemma 2.2. *Let* $\{a_n\}$ *be a sequence in* $P \cap U$. *Then the element* $a = \sum_{n=1}^{\infty} a_n$ *is in* U *and* $I(a) = \sum_{n=1}^{\infty} I(a_n)$

Proof. By the Lemma 2.1 for each n there is a sequence $\{b_{n,k}\}$ in $P \cap L$ such that $a_n = \sum_{k=1}^{\infty} b_{n,k}$. Hence $a = \sum_{k,n} b_{n,k}$. Since the set of pairs of integers is countable, a is the sum of a sequence in $P \cap L$ and so must be in U. Also $I(a) = \sum_{k,n} I(b_{n,k}) = \sum_{n=1}^{\infty} I(a_n)$

Lemma 2.3. *Let* $\{e_k\}$ *be a sequence in* P, *and let* $e = \sum_{k=1}^{\infty} e_k$. *Then* $\bar{I}(e) \leqq \sum_{k=1}^{\infty} \bar{I}(e_k)$.

Proof. If $\bar{I}(e_k) = \infty$ for some k, we are done. If not, given $\varepsilon > 0$, there is an element $j_k \in U$ such that $e_k \leqq j_k$ and $I(j_k) \leqq \bar{I}(e_k) + \varepsilon \cdot 2^{-k}$. Since each j_k is in P, Lemma 2.2 implies that the element $j = \sum_{k=1}^{\infty} j_k$ is in U and that $I(j) = \sum_{k=1}^{\infty} I(j_k) \leqq \sum_{k=1}^{\infty} \bar{I}(e_k) + \varepsilon$. Since $j \geqq e$, we have $\bar{I}(e) \leqq \sum_{k=1}^{\infty} \bar{I}(e_k) + \varepsilon$, and the Lemma follows since ε was an arbitrary positive number.

Lemma 2.4. *Let* $\{e_n\}$ *be an increasing sequence of elements in* $S(I)$, *and let* $e = \lim e_n$. *Then* $e \in S(I)$ *if and only if* $\lim I(e_n) < \infty$. *In this case* $I(e) = \lim I(e_n)$.

Proof. Since $e \geqq e_n$, $\bar{I}(e) \geqq I(e_n)$. Thus if $\lim I(e_n) = \infty$, then $\bar{I}(e) = \infty$, and $e \notin S(I)$. Suppose $\lim I(e_n) < \infty$. Set $j = e - e_1$. Then $j \geqq 0$, and $j = \sum_{n=1}^{\infty} (e_{n+1} - e_n)$. Hence by Lemma 2.3 $\bar{I}(j) \leqq \sum_{n=1}^{\infty} I(e_{n+1} - e_n) = \sum_{n=1}^{\infty} \{I(e_{n+1}) - I(e_n)\} = \lim I(e_n)$

$-I(e_1)$. Thus $\bar{I}(e)=\bar{I}(e_1+j)\leqq I(e_1)+\bar{I}(j)\leqq\lim I(e_n)$. Since $e_n\leqq e$, we have $\underline{I}(e)\geqq I(e_n)$, and so $\underline{I}(e)\geqq\lim I(e_n)$. Thus $\underline{I}(e)=\bar{I}(e)=I(e)=\lim I(e_n)$.

Theorem 2.5. *Let $\{e_k\}$ be a sequence in $P\cap S(I)$. Then the element $\inf e_k$ is in $S(I)$, and the element $\underline{\lim}\ e_k$ is in $S(I)$ if $\underline{\lim}\ I(e_k)<\infty$. In this case $I(\underline{\lim}\ e_k)\leqq\underline{\lim}\ I(e_k)$.*

Proof. Let $j_n=e_1\wedge e_2\wedge\cdots\wedge e_n$. Then $\{j_n\}$ is a sequence in $P\cap S(I)$, which decrease to $j=\inf e_k$. Thus $-j_n\uparrow -j$, and since $I(-j_n)=-I(j_n)\leqq 0$, we must have $j\in S(I)$ by Lemma 2.4. To prove the rest of the theorem, let $h_n=\inf\{e_k : k\geqq n\}$. Then $\{h_n\}$ is a sequence in $P\cap S(I)$ which increase to $\underline{\lim}\ e_k$. Since $h_n\leqq e_k$ for $n\leqq k$, $\lim I(h_n)\leqq\underline{\lim}\ I(e_k)<\infty$. Hence $\underline{\lim}\ e_k\in S(I)$ and $I(\underline{\lim}\ e_k)\leqq\underline{\lim}\ I(e_k)$ by Lemma 2.4.

Theorem 2.6. *Let $\{e_n\}$ be a sequence of elements in $S(I)$ and suppose that there is an element j,k in $S(I)$ such that for all n we have $j\leqq e_n\leqq k$. If $e=\lim e_n$, then e is an element in $S(I)$ and we have $I(e)=\lim I(e_n)$*

Proof. The element $k-e_n$ are in P, and $I(k-e_n)\leqq 2I(k)$. Hence by Theorem 2.5 we have $k-e$ in $S(I)$ and $I(k-e)\leqq\underline{\lim}\ I(k-e_n)=I(k)-\overline{\lim}\ I(e_n)$. Hence $\overline{\lim}\ I(e_n)\leqq I(e)$. Since the element e_n-j are also in P, we have $I(e-j)\leqq\overline{\lim}\ I(e_n-j)=\underline{\lim}\ I(e_n)-I(j)$. Hence $I(e)\leqq\underline{\lim}\ I(e_n)$, and so $\lim I(e_n)$ exists and is equal to $I(e)$.

III. I-measurability on $S(I)$

We now turn our attention to the I-measurability on the I-summable class $S(I)$.

Definition 3.1. For an element a in E, a is said to be *I-measurable* if $a\wedge e$ is in $S(I)$ for each e in L. The class of I-measurable elements will be denoted by $M(I)$. It is clear that if $a\in P$ and $a\wedge e\in S(I)$ for every $e\in P\cap L$ then $a\in M(I)$. Indeed $a\wedge e=(a\wedge e^+)-e^-$ whenever $a\geqq 0$.

Lemma 3.2. *Let a is in $M(I)$, then $a\wedge s$ is in $S(I)$ for each s in $S(I)$*

Proof. Let $a\in M(I)$, $s\in S(I)$ and suppose $\varepsilon>0$ is given. Choose $r\in -U$ and $t\in U$ such that $r\leqq s\leqq t$ and so that $I(t)-I(r)<\varepsilon$. We can find $t_n\in L$ and $r_n\in L$ for which $t_n\uparrow t$, $r_n\downarrow r$ and $\lim I(t_n)=I(t)$, $\lim I(r_n)=I(r)$. Then $a\wedge t_n\in S(I)$, $a\wedge r_n\in S(I)$, and $\{a\wedge t_n\}$ is a sequence of elements in $S(I)$ converging to $a\wedge t$. Since $a\wedge r_k\leqq a\wedge t_n\leqq t_n$ for fixed k. Thus by Theorem 2.6, we have $a\wedge t\in S(I)$. Since $a\wedge s=a\wedge(t\wedge s)=(a\wedge t)\wedge s$, we have $a\wedge s\in S(I)$.

Theorem 3.3. *$M(I)$ is an integration lattice.*

Proof If $a,b\in M(I)$ and $e\in L$, then $a\wedge e\in S(I)$, $b\wedge e\in S(I)$ and $(a\vee b)\wedge e=(a\wedge e)\vee(b\wedge e)$, $(a\wedge b)\wedge e=(a\wedge e)\wedge(b\wedge e)$. Thus by Theorem 1.4, $(a\vee b)\wedge e\in S(I)$, and $(a\wedge b)\wedge e\in S(I)$, so $a\vee b$ and $a\wedge b$ are in $M(I)$. Consequently, $M(I)$ is a lattice. Let $a,b\in M(I)$ and k is a real number. If $e\in L$ and $k\neq 0$, then $(ka)\wedge e=k(a\wedge\frac{e}{k})$ belongs to $S(I)$ and so $ka\in M(I)$. If $k=0$, then $ka\in M(I)$: for $ka=0$. Let a, $b\in M(I)$, we have $a+b=(a^++b^+)-(a^-+b^-)$. Since $a^-=a\wedge 0$, $b^-=b\wedge 0$, by Theorem 1.4 it follows that a^-+b^- is in $S(I)$. Let $u\in L$ and $u\geqq 0$, we obtain $(a^++b^+)\wedge u=(a^+\wedge u+b^+\wedge u)\wedge u$. Since the right side belongs to $S(I)$, we have $a^++b^+\in M(I)$. Given $e\in L$, we have $(a+b)\wedge e=\{(a^++b^+)-(a^-+b^-)\}\wedge e=(a^++b^+)\wedge(a^-+b^-+e)-(a^-+b^-)$. Since $a^++b^+\in M(I)$ and $a^-+b^-\in S(I)$, it follows from Lemma 3.2 that $(a+b)\wedge e\in S(I)$. Therefore $a+b\in M(I)$. Finally, since $L\subset M(I)$, second condition of integration lattices holds.

Lemma 3.4. *Let $\{a_n\}$ be a sequence of element in $M(I)$, and let $b\in M(I)$ such that $a_n\geqq b$, for $n=1,2,3,\cdots$. If $a=\lim a_n$, then a is an element in $M(I)$.*

Proof. If $e\in L$, then $a_n\wedge e\in S(I)$, $b\wedge e\in S(I)$ and $\{a_n\wedge e\}$ is a sequence of elements in $S(I)$ converging to $a\wedge e$. Since $(b\wedge e)\leqq(a_n\wedge e)\leqq e$, for $n=1,2,3,\cdots$. Thus by Theorem 2.6, we have $a\wedge e\in S(I)$. Hence $a\in M(I)$

Theorem 3.5. *Let $\{a_n\}$ be a sequence of elements in $M(I)$, then $\sup a_n$ is an element in $M(I)$. In particular, if $a_n \geqq e$, $n=1,2,3,\cdots$, for some $e \in M(I)$, then $\inf a_n$, $\underline{\lim}\ a_n$, and $\overline{\lim}\ a_n$ belong to $M(I)$.*

Proof. Let $b_n = a_1 \vee a_2 \vee \cdots \vee a_n$. Then $\{b_n\}$ is a sequence in $M(I)$, which increase to $b = \sup a_n$ and $b_n \geqq a_1$. Thus by Lemma 3.4. $\sup a_n$ is an element in $M(I)$. To prove the rest of the theorem, let $c_n = a_1 \wedge a_2 \wedge \cdots \wedge a_n$. Then $\{c_n\}$ is a sequence in $M(I)$, which decrease to $c = \inf a_n$ and $c_n \geqq e$. Thus by Lemma 3.4. $\inf a_n$ is an element in $M(I)$. Since $\underline{\lim}\ a_n = \sup_n (\inf_{k \geqq n} a_k)$ and $\overline{\lim}\ a_n = \inf_n (\sup_{k \geqq n} a_k)$, the proof is completed.

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