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A Note on *H* Cogroups

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⟨Abstract⟩

In this paper we shall introduce the concept of *H* cogroup, dual to that of *H* group and construct the example of *H* cogroup by dualiging the loop space.

H Cogroup에 관하여

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⟨요 약⟩

H group의 쌍대인 *H* cogroup의 개념을 소개하고, loop space를 쌍대화함으로써 *H* cogroup의 예를 제시한다.

I. Introduction

It is well known that the *H* spaces(or *H* groups) play an important role in the theory of homotopy. The loop space of a pointed topological space (Y, y_o), denoted by ΩY, is the useful example of an *H* space.

In this note we shall try to introduce the concept of *H* cogroup, dual to that of *H* group and construct the example of *H* cogroup by dualiging the loop space.

II. *H* cogroups

Let C be the category of pointed topological spaces with the continuous mappings preserving the base points. If (X, x_o) and (Y, y_o) are the objects C, Their sum $X \vee Y = X \times \{y_o\} \cap \{x_o\} \times Y$ regarded as the subspace of $X \times Y$ is also an object of C with the base point (x_o, y_o). The morphism $X \vee Y$ to Z is defined as follows. If $f: X \to Z$ and $g: Y \to Z$ are the base point preserving continuous mappings, we let $(f, g): X \vee Y \to Z$ be that map defined by $(f,g)|X=$ and $(f,g)|Y=g$, the (f,g) is also a base point preserving continuous map.

An *H* cogroup consists of a pointed topological space **Q** together with a continuous comultiplication.

$$\lambda : Q \to Q \vee Q$$

such that the following properties 1)~3) hold.

1) If $c: Q \to Q$ is the constant map, then each composite map, then each composite map

$$Q \xrightarrow{\lambda} Q \vee Q \xrightarrow{(c,1)} Q \text{ and } Q \xrightarrow{\lambda} Q \vee Q \xrightarrow{(1,c)} Q$$

is homotopic to 1Q. (Existence of homotopy identity)

2) The square diagram

$$\begin{array}{ccc} Q & \xrightarrow{\lambda} & Q \vee Q \\ \lambda \downarrow & & \downarrow \lambda \vee 1 \\ Q \vee Q & \xrightarrow[1 \vee \lambda]{} & Q \vee Q \vee Q \end{array}$$

is homotopic commutative, that is, $(\lambda \vee 1) \circ \lambda \cong$

$(1\vee\lambda)\circ\lambda$. (where if f: $X\to X'$. g: $Y\to Y'$. map $f\vee g: X\vee Y\to X'\vee Y'$ is defined by $(f\vee g)(x_o, y) = (x_o', g(y))$ (Homotopy associativity).

3) There exists a map $\phi: Q\to Q$ such that each composite map

$$Q\xrightarrow{\lambda}Q\vee Q\xrightarrow{(1,\phi)}Q \text{ and } Q\xrightarrow{\lambda}Q\vee Q\xrightarrow{(\phi,1)}Q$$

is homotopic to $c: Q\to Q$. (Existence of homotopy inverse) An H cogroup is said to be abelian if the triangle

$$\begin{array}{ccc} & Q & \\ \swarrow^{\lambda} & & \searrow^{\lambda} \\ Q\vee Q & \xrightarrow{T'} & Q\vee Q \end{array}$$

where $T'(q_1, q_2) = (q_2, q_1)$ for $q_1, q_2 \in R$, is homotopic commutative.

If Q and Q' are H cogroups with comultiplications λ and λ' respectively, a continuous map $\beta: Q\to Q'$ is called a homomorphism if the square diagram

$$\begin{array}{ccc} Q & \xrightarrow{\lambda} & Q\vee Q \\ \beta\downarrow & & \downarrow\beta\vee\beta \\ Q' & \xrightarrow[\lambda']{} & Q'\vee Q' \end{array}$$

is homotopic commutative.

III. Example of an *H* cogroup

Let Z be a pointed topological space with the base point z_o. The suspension of Z, denoted by SZ, is the quotient space of $Z\times I$ in which $(Z\times 0)\cup(z_o\times I)\cup(Z\times 1)$ has heen identified to a single point which is the base point of SZ, where $I=[0, 1]$.

If $(z,t)\varepsilon Z\times I$, we use $[z,t]$ to denote the corresponding point of SZ under the quotient map $Z\times I\to SZ$. Then $[z,0]=[z_o, t]=[z', 1]$ for all z, z' εZ ant t ε I. we also denote the point $[z_o, 0]\varepsilon SZ$ by Z_o.

Theorem 1. If we define a comultiplication

$$\lambda: SZ\to SZ\vee SZ$$

by the formula

$$\lambda([z, t]) = \begin{cases} ([z, 2t], z_o) & 0\leqq t\leqq \frac{1}{2} \\ (z_o, [z, 2t-1]) & \frac{1}{2}\leqq t\leqq 1 \end{cases}$$

then SZ is an H cogroup.

(proof) It is easily seen that the function λ defined is above is welled defined because of, for $t=\frac{1}{2}$, $([z, 2t], z_o)=([z,1], z_o)=(z_o, z_o)$ and $(z_o, [z, 2t-1])=(z_o, [z,0])=(z_o, z_o)$.

we now prove that SZ in an H-cogroup.

1) If $c: SZ\to SZ$ in the constant map then

$$(c, 1)\circ\lambda([z, t]) = \begin{cases} z_o & 0\leqq t\leqq \frac{1}{2} \\ [z, 2t-1] & \frac{1}{2}\leqq t\leqq 1 \end{cases}$$

Now let $F_1: SZ\times I\to SZ$ be a map defined by

$$F_1([z,t], t') = \begin{cases} \left[z, \frac{2tt'}{t'+1}\right], & 0\leqq t'\leqq 2t-1 \\ \left[z, \frac{2t-1+t'^2}{t'+1}\right], & 2t-1\leqq t'\leqq 1 \end{cases}$$

then F_1 is a well defined since $\left[z, \frac{2tt'}{t'+1}\right]=[z, t']$ and $\left[z, \frac{2t-1+t'^2}{t'+1}\right]=[z, t']$ when $t'=2t-1$ and it is easilyseen that F_1 is the homotopy between $(c, 1)\circ\lambda$ and 1_{SZ}. We can also see that the function $F_2: SZ\times I\to SZ$ defined by

$$F_2([z,t], t') = \begin{cases} \left[z, \frac{2t}{t'+1}\right], & 0\leqq t'\leqq 2t-1 \\ \left[z, \frac{2tt'-t'+1}{t'^2+1}\right], & 2t-1\leqq t'\leqq 1 \end{cases}$$

is the homotopy between $(1, c)\circ\lambda$ and 1_{SZ}.

2) We have $(\lambda\vee 1)\circ\lambda([z,t])$

$$= \begin{cases} (([z, 4t], z_o), z_o) & 0\leqq t\leqq \frac{1}{4} \\ ((z_o, [z, 4t-1]), z_o) & \frac{1}{4}\leqq t\leqq \frac{1}{2} \\ (z_o, (z_o, [z, 2t-1])) & \frac{1}{2}\leqq t\leqq 1 \end{cases}$$

$(1\vee\lambda)\circ\lambda([z,t])$

$$= \begin{cases} ([z, 2t], z_o), z_o) & 0\leqq t\leqq \frac{1}{2} \\ (z_o, ([z, 4t-2], z_o)) & \frac{1}{2}\leqq t\leqq \frac{3}{4} \\ (z_o, (z_o, [z, 4t-3])) & \frac{3}{4}\leqq t\leqq 1 \end{cases}$$

Now let F: $SZ\times I\to SZ\vee SZ\vee SZ$ a function defined bo

$F([z,t], t')$

$$= \begin{cases} \left(\left(\left[z, \frac{4t}{1+t'}\right], z_o\right), z_o\right) & 0\leqq t\leqq \frac{1}{4}(t'+1) \\ (z_o, ([z, 4t-t'-1], z_o)) & \frac{1}{4}(t'+1)\leqq t \leqq \frac{1}{4}(t'+2) \\ \left(z_o, \left(z_o, \left[z, \frac{4t-t'-2}{2-t'}\right]\right)\right) & \frac{1}{4}(t'+2)\leqq t\leqq 1 \end{cases}$$

than F is well defined and continuous, therefore the homotopy between $(\lambda\vee 1)\circ\lambda$ and $(1\vee\lambda)\circ\lambda$, That is, $(\lambda\vee 1)\circ\lambda\cong(1\vee\lambda)\circ\lambda$

3) Let $\varphi: SZ\to SZ$ be a function defined by

$$\varphi([z,t])=[z,1-t]$$

than we have

$$(1,\varphi)\circ\lambda([z,t])=\begin{cases}([z,2t],\ z_o) & 0\leqq t\leqq\frac{1}{2}\\ (z_o,\ [z,\ 2-2t]) & \frac{1}{2}\leqq t\leqq 1\end{cases}$$

Now let $F: SZ\times I\to SZ$ be a function defined by

$$F([z,t],\ t')=\begin{cases}([z,\ (2t-2t')(1-t')],\ z_o) & 0\leqq t\leqq\frac{t'+1}{2}\\ (z_o,\ [z,\ (2-2t)(1-t')]) & \frac{t'+1}{2}\leqq t\leqq 1\end{cases}$$

then F is well defined and continuous, therefore the homotopy between $(1,\ \varphi)\circ\lambda$ and the constant function $c: SZ\to SZ$. Similary we can show that $(\varphi,\ 1)\circ\lambda\simeq c$.

By 1), 2) and 3) SZ is an H cogroup.

If $f: Z\to Z'$ then we define

$$Sf: SZ\to SZ'$$

by $Sf([z,t])=[f(z),\ t]$.

Now let SZ and SZ' be H cogroups with comultiplication λ and λ' respectively, and $f: Z\to Z'$ then we have

$$(Sf\vee Sf)\circ\lambda([z,t])=\begin{cases}([f(z),2t],z_o) & 0\leqq t\leqq\frac{1}{2}\\ (z_o,[f(z),2t-1]) & \frac{1}{2}\leqq t\leqq 1\end{cases}$$

and $\lambda'\circ Sf([z,\ t])=\begin{cases}([f(z),\ 2t],\ z_o) & 0\leqq t\leqq\frac{1}{2}\\ (z_o,[f(z),2t-1]) & \frac{1}{2}\leqq t\leqq 1\end{cases}$

This shows that the square diagiam

$$\begin{array}{ccc} SZ & \xrightarrow{\lambda} & SZ\vee SZ\\ {\scriptstyle S_f}\downarrow & & \downarrow{\scriptstyle S_f\vee S_f}\\ SZ & \xrightarrow{\lambda'} & SZ'\vee SZ'\end{array}$$

is commutative. Hence Sf is a homomorphism. From the above result we have the following.

Theorem 2, S is a covariant functor from the category C of pointed topological spaces and continuous maps to the category D of H cogroups and homomorphisms.

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