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On the continuity in a bitopological space

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〈Abstract〉

In a bitopological space, we define a pairwise-continuous map and a quasi-continuous map. And we find their properties, the relations between them. We get the continuity of quasi T_1 space, that is, the image of a quasi T_1 space under a quasi-continuous map is quasi T_1.

쌍 위상공간상의 연속성에 대하여

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〈요 약〉

쌍 위상공간상에서 우리는 새로운 pairwise연속인 함수와 quasi연속인 함수를 정의한다. 그리고 그들의 성질과 그들 상호관계를 알아보고, 또 quasi T_1공간의 연속성에 관한 정리를 얻는다.

I. Introduction

A subset S of a bitopological space (X, P, Q) is quasiopen [1] if for every $x \in S$, there exists a P-open set U such that $x \in U \subset S$, or a Q-open set V such that $x \in V \subset S$. It is clear that a quasiopen set in a space (X, P, Q) is a union of a P-open set and a Q-open set [1]. Any union of quasiopen sets is also quasiopen. Let A be a quasiopen set and CA denote the complement of A, then CA is termed quasiclosed [1]. Generally in a topological space, we want to relate different topological spaces. For given topological spaces $(X, T_X)$and (Y, T_Y), a map $f: X \to Y$ relates sets and $f^{-1}: T_Y \to T_X$. And we defind that let X and Y be spaces, a map $f: X \to Y$ is said to be continuous if the inverse image of each open set in Y is open in X; that is, f^{-1} maps T_Y into T_X.

Similarily we shall define a continuity of a map in a bitopological space. The purpose of this paper is to study some properties of quasiopen sets and the continuity of a map relating different bitopological spaces.

In this paper for a arbitray set X, P_X and Q_X will denote a P-open set and a Q-oen set in X respectively. And CA denote the complement of A.

II. Quasiopen sets and spaces, pairwise T_1 and qusi T_1

Datta [1] remarked that the intersection of two quasiopen sets may not be quasiopen. But in theorem 1 [3], the sufficient condition to be quasiopen was shown. Let (Y, P_Y, Q_Y) be a subspace of a space (X, P_X, Q_X). If A is quasiopen in Y and Y is even if P-open (resp. Q-open) A may not be quasiopen in X. For,

Example 1. Let $X=\{a, b, c, d\}$, $P=\{\phi, \{c, d\}, X\}$ and $Q=\{\phi, \{a, d\}, X\}$. Let $Y=\{a, d\}$. Then $\{d\}$ is quasiopen in the subspace Y of X, but it is not quasiopen in X.

However, in [3] they showed that let (Y, P_Y, Q_Y) be a biopen subspace of (X, P_X, Q_X),

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then a quasiopen set A in Y is quasiopen in X.

The term pairwise T_1 is due to Murdeshwar and Naimpally [2].

Definition 1. A space (X, P_X, Q_X) is pairwise T_1 [2] if for x, $y \in X$, $x \neq y$, there exist a P-open set U and a Q-open set V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

And also, a quasi T_1 is due to Maheshwari, Jain and Chae [3].

Definition 2. A space (X, P_X, Q_X) is termed quasi T_1 if for x, $y \in X$, $x \neq y$, there exist quasiopen sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

In [3], they got that every pairwise T_1 space is quasi T_1 and every subspace of a quasi T_1 space is quasi T_1. And also they got,

Proposition 1: A space (X, P_X, Q_X) is quasi T_1 if and only if singletons are quasiclosed.

Proof. Confer the theorem 5 in [3].

III. Continuity in a bitopological space

we shall define a map and relate sets, or spaces. For given bitopological spaces (X, P_X, Q_X) and (Y, P_Y, Q_Y), we can consider a map $f: X \to Y$ relating sets and $f^{-1}: P_Y \to P_X$, or f^{-1}: $Q_Y \to Q_X$, or $f^{-1}: P_Y \cup Q_Y \to P_X \cup Q_X$ where P_X and Q_X are P-open sets and Q-open sets in X, P_Y and Q_Y are P-open sets and Q-open sets in Y respectively.

Definition 3. Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological space. A map $f: X \to Y$ is said to be pairwise-continuous if the inverse image of each set of P-open sets in Y is P-open in X and that of each set of Q-open sets in Y is Q-open in X.

We can define a map to be weaker than a pairwise-continuous map in definition 3.

Definition 4. Let (X, P_X, Q_Y) and (Y, P_Y, Q_Y) be bitopological spaces. A map $f: X \to Y$ is said to be quasi-continuous if the inverse image of each set of quasiopen sets in Y is quasiopen in X.

Example 2. Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. A constant map $f: X \to Y$ is always quasi-continuous: the inverse imae of any quasiopen set U in Y is either ϕ or X, which are quasiopen in X.

We have,

Proposition 2: A pairwise-continuous map is quasi-continuous.

Proof. Since every quasiopen set is a union of a P-open set and a Q-open set, and also any union of quasiopen sets is quasiopen, the proof is obvious.

Example 3. Let $X = \{a, b, c\}$, $P = \{\phi, \{a\}, X\}$ and $Q = \{\phi, \{c\}, X\}$. Then the set $\{a, c\}$ is quasiopen but it is neither P-open nor Q-open. From example 3, it is clear that the converse of proposition 2 is not true.

Proposition 3: Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. A map $f: X \to Y$ is quasi-continuous if and only if the inverse image of every quasiclosed subset of Y is a quasiclosed subset of X.

Proof. Suppose f is quasi-continuous and let F be a quasiclosed subset of Y. Then the complement of F is quasiopen, and so $f^{-1}(CF) = Cf^{-1}(F)$ is quasiopen in X. Hence $f^{-1}(F)$ is quasiclosed in X. Conversely let A be a quasiopen set in Y. And let $B = Y - A$; then B is quasiclosed in Y. By hypothesis, $f^{-1}(B)$ is a quasiclosed subset of X. In elementary set theory, $f^{-1}(A) = f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$, so $f^{-1}(A)$ is quasiopen in X. Hence f is quasi-continuous.

The continuity as we have defined is a global property, that is, it restricts the way in which a map behaves on the entire set. There also exists a corresponding local concept of continuity at a point.

Defintion 5. Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. A map $f: X \to Y$ is said to be pairwise-continuous at p in X if the inverse image of every P-open subset of Y containing $f(p)$ is P-open in X and that of every Q-open subset of Y containing $f(p)$ is Q-open in X.

Similarily,

Definition 6. Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. A map $f: X \to Y$ is said to be quasi-continuous at p in X if the inverse image of every quasiopen subset of Y containing $f(p)$ is quasiopen in X.

From the above definitions we have

Proposition 4: Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. Then a map $f: X \to Y$ is pairwise-continuous if and only if it is pairwise-continuous at every point of X.

Proof. Suppose f is pairwise-continuous and let U be a P-opeset in Y containing $f(p)$, then $p \in f^{-1}(U)$ and $f^{-1}(U)$ is P-open in X. If a Q-open set of Y, V contains $f(p)$, then $p \in f^{-1}(V)$ and $f^{-1}(V)$ is Q-open in X. So f is pairwise-continuous at $p \in X$. Conversely, assume f is pairwise-continuous at every point p of X and let U and V be P-open and Q-open in Y respectively. Then for every $p \in f^{-1}(U)$, there exists a P-openset A_p in X such that $p \in A_p \subset f^{-1}(U)$. If $p \in f^{-1}(V)$, there exists a Q-openset B_p in X such that $p \in B_p \subset f^{-1}(V)$. Since $f^{-1}(U)$ is a union of $\{A_p\}$ and also $f^{-1}(V)$ that of $\{B_p\}$, f is pairwise-continuous.

Proposition 5: Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. Then a map $f: X \to Y$ is quasi-continuous if and only if it is quasi-continuous at every point of X.

Proof. Assume f is quasi-continuous and let $H \subset Y$ be a quasi-spen set containing $f(p)$. Then $p \in f^{-1}(H)$ and $f^{-1}(H)$ is quasiopen in X. Hence f is quasi-continuous at p of X. Conversely, suppose f is quasi-continuous at every point $p \in X$. Let $H \subset Y$ be a quasiopen set. Then for $p \in f^{-1}(H)$, there exists a quasiopen set $G_p \subset X$ such that $p \in G_p \subset f^{-1}(H)$. And $f^{-1}(H) = \cup \{G_p : p \in f^{-1}(H)\}$ is a union of quasiopen sets. Accordingly $f^{-1}(H)$ is quasiopen, so f is quasi-continuous.

From proposition 2, 4 and 5, we get,

Corollary 1: Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. If a map $f: X \to Y$ is pairwise-continuous at point p in X, then f is quasi-continuous at point p in X.

We know that in a topological space, if f and g are continuous maps then the composition of f and g, $g \circ f$ is also continuous.

Similaryly in a bitopolgical space,

Proposition 6: Let (X, P_X, Q_X), (Y, P_Y, Q_Y) and (Z, P_Z, Q_Z) be bitopological spaces. If $f: X \to Y$ and $g: Y \to Z$ are pairwise-contiuous, so also is $g \circ f: X \to Z$ where $g \circ f$ is the composition of f and g.

Proop. If U is P-open in Z, then $g^{-1}(U)$ is P-open in Y and $f^{-1}(g^{-1}(U))$ is P-open in X. By elementary set theory, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, so $(g \circ f)^{-1}(U)$ P-open in X. And V is Q-open in Q-open in Z, then $g^{-1}(V)$ is Q-open in Y. And $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is Q-open in X. Accordingly $g \circ f$ is pairwise-continuous.

From proposition 2 and 6 we have.

Corollary 2: Let (X, P_X, Q_X), (Y, P_Y, Q_Y) and (Z, P_Z, Q_Z) be bitopological spaces. If $f: X \to Y$ and $g: Y \to Z$ are quasi-continuous maps, so also is $g \circ f: X \to Z$.

We get the followings from proposition 4, 5, 6, corollary 1 and 2.

Corollary 3: Let (X, P_X, Q_X), (Y, P_Y, Q_Y) and (Z, P_Z, Q_Z) be bitopological spaces. If $f: X \to Y$ and $g: Y \to Z$ are pairwise-continuous at $p \in X$, so also is $g \circ f: X \to Z$.

Corollary 4: Let (X, P_X, Q_X), (Y, P_Y, Q_Y) and (Z, P_Z, Q_Z) be bitopological spaces. If $f: X \to Y$ and $g: Y \to Z$ are quasi-continuous at $p \in$ X, so also is $g \circ f: X \to Z$.

We shall consider the continuity of quasi T_1 space [3].

Proposition 7: Let (X, P_X, Q_X) and (Y, P_Y, Q_Y) be bitopological spaces. If a map f : $X \to Y$ is quasi-continuous and $f(X)$ is taken the subspace $(f(X), P_{f(X)}, Q_{f(X)})$, then f: $X \to f(X)$ is quasi-continuous.

Proof. Let U be a quasiopen subset of Y.

Then $f^{-1}(U \cap f(X))=f^{-1}(U) \cap f^{-1}(f^{-1}(U)$ is quasiopen subset of X.

Theorem 1: The image of a quasi T_1 space under a quasi-continuous map is quasi T_1.

Proof. Let (X, P_X, Q_X) be quasi T_1 space and a map $f: X \rightarrow Y$ quasi-continuous. We must to show that $f(X)=\{z : z=f(p), p \in X\}$ is quasi T_1 space. By propoition 5 and 7, a map $f: X \rightarrow f(X)$ is quasi-cotinuous if f is quasicontinuous at every point of X, that is, f is quasi-continuous at x and also at y for $x, y \in X$. Accordingly for $z'=f(x)$, $z''=f(y) \in f(X)$, there exist quasiopen sets of $f(X)$, H and G such that for $z' \in H$, $z'' \in G$, $x \in f^{-1}(H)$ and $y \in f^{-1}(G)$ are quasiopen in X and for $x, y \in X$, $x \neq y$, $x \in f^{-1}(H)$, $y \notin f^{-1}(H)$ and $x \notin f^{-1}(G)$, $y \in f^{-1}(G)$ since X is quasi T_1 space. This must be made like that for $z' \neq z''$, $z'=f(X) \in H$, $z''=f(y) \notin H$ and $z'=f(X) \notin G$, $z''=f(y) \in G$. It is evident that $z'=f(x) \neq f(y)=z''$. For, suppose $f(x)=f(y)$, there are not quasiopen sets of X, U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. This contradicts that X is quasi T_1 space.

From proposition 1 and theorem 1, we get,

Corollary 5: The singletons of a quasi T_1 space under a quasi-continuous map are quasiclosed.

We are sure that this paper shall contribute to the study of relations among different bitopological spaces by defining suitably open map and closed map, compactness, uniformity and etc.

OPEN QUASTION:

The image of a paiwise T_1 space under a quasi-continuous map is pairwise T_1 or quasi T_1?

The image of a quasi T_1 space under a pairwise-continuous map is quasi T_1 or pairwise T_1?

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