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A Notice on the Extension of Modules

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⟨Abstract⟩

Let R be a left Artinian ring, and $J(R)$ the Jacobson radical of R. The exact sequence

$$\cdots\cdots\longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow A \longrightarrow 0$$

is a projective resolution of an R-module A. Consider the sequence

$$0 \longrightarrow \mathrm{Hom}_R(P_0, C) \xrightarrow{d_1'} \mathrm{Hom}_R(P_1, C) \xrightarrow{d_2'} \mathrm{Hom}_R(P_2, C) \longrightarrow\cdots\cdots,$$

for an R-module C. Then the 1st homology group Ker d_2'/Im d_1' is denoted by $\mathrm{Ext}^1_R(A, C)$. In this note, we show the following characterization. If C is an R-module and $J(R)C=0$ then C is semi-simple, where $J(R)C=\left\{\sum_{i=1}^{n} r_i c_i \mid r_i \in J(R),\ c_i \in C\right\}$. And we also prove, in this case, that $\mathrm{Ext}^1_R(A, C)=0$ provided that $\mathrm{Ext}^1_R(A, S)=0$ for some simple R-module S, when A is an R-module.

Module의 확장에 대하여

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⟨요 약⟩

R은 left Artinian ring이라하고, $J(R)$을 R의 Jacobson radical 이라두자. 우리는 R-module A의 projective resolution을

$$\cdots\cdots\longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow A \longrightarrow 0$$

로 놓고, 어떤 R-module C에 대하여 다음과 같은 sequence를 생각한다.

$$0 \longrightarrow \mathrm{Hom}_R(P_0, C) \xrightarrow{d_2'} \mathrm{Hom}_R(P_1, C) \xrightarrow{d_1'} \mathrm{Hom}_R(P_2, C) \longrightarrow\cdots\cdots.$$

이때 the 1st homology group Ker d_2'/Im d_1'을 $\mathrm{Ext}^1_R(A, C)$로 나타내자. 그러면 본 논문에서는 다음과 같은 성질이 얻어진다. 만일 C가 R-module로서 $J(R)C=0$이면, C는 semi-simple 임을 말 할 수 있는데, 여기서 $J(R)C$는 다음과 같은 모든 유한화의 집합을 나타낸다.

$$J(R)C=\left\{\sum_{i=1}^{n} r_i c_i \mid r_i \in J(R),\ c_i \in C\right\}.$$

또 이 경우에 A가 하나의 R-module 일때, 어떤 simple R-module S에 대하여 $\mathrm{Ext}^1_R(A, S)=0$ 이면, $J(R)C=0$가 되는 여하한 R-module C에 대해서도, $\mathrm{Ext}^1_R(A, C)=0$ 임을 밝힌다.

I. Introduction

Throughout this paper, every ring has a non-zero identity, and every module is unitary without otherwise specified. $J(R)C$ is the set of all finite sums of the products by the pairs of elements of the radical $J(R)$ and the module C.

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By A, we denote an R-module. The main purpose of this paper is the following fact; the condition $J(R)C=0$ for an R-module C implies $\mathrm{Ext}^1_R(A,C)=0$, when $\mathrm{Ext}^1_R(A,S)=0$ for some simple R-module S.

In Section Ⅱ, some necessary terminologies and lemmas are introduced. The Lemma 3 of this section tells us that the condition $J(R)C=0$ assures the semi-simplicity of the given module C, where R is a left Artinian ring.

In Section Ⅲ, the main result which is mentioned above is obtained. And we give an information about the projective module in our discussion.

Ⅱ. Preliminaries

We recall that a ring R is called left Artinian if R has the minimum condition on left ideals, namely, the left R-module $_RR$ is left Artinian. And a ring R is called semi-simple in case the R-module R is semi-simple, that is, R can be represented as a direct sum of left ideals which are simple, considered as R-modules.

It is well-known that the Jacobson radical $J(R)$ of R is the intersection of all maximal left ideals of R [1].

Before characterizing our main goal of this note, we need the following lemmas which can be found in [2].

Lemma 1

If R is a left Artinian ring, then the factor ring $R/J(R)$ is semi-simple and left Artinian.

Lemma 2

Every non-zero R-module is the direct sum of simple R-submodules if and only if R is the direct sum of a finite number of simple left ideals;

$$R=\bigoplus\sum_{i=1}^{n} L_i,$$

where each L_i is a simple left ideal and $L_i=Re_i$, where $\{e_i\}^n_{i=1}$ is a set of orthogonal idempotents such that $\sum_{i=1}^{n} e_i=1$. That property of Lemma 2 is sometimes called the Structure Theorem for Semi-simple Rings with Minimum Condition.

Using the Lemma 1 and Lemma 2, we can deduce the following fact which will be used in verifying our main result.

Lemma 3

Let R be a left Artinian ring, and $_RC$ an left R-module. If $J(R)C=0$, then C is the direct sum of simple R-submodules.

Proof. Since R is a left Artinian ring, the factor ring $R/J(R)$ is, by Lemma 1, a semi-simple ring which is left Artinian.

Now, define an operation $\sigma : R/J(R)\times C \longrightarrow C$ with $\sigma(r+J(R),\ c)=rc,\ r\in R,\ c\in C$. Then σ is well-defined by the assumption $J(R)C=0$. And it can be easily seen that C is a left $R/J(R)$-module.

From this, $R/J(R)$ is semi-simple left Artinian, and by Lemma 2, $_{R/J(R)}C$ can be represented as a direct sum of simple $R/J(R)$-submodules such that

$$C=\bigoplus\sum_{\alpha} C_\alpha,$$

where C_α is a simple $R/J(R)$-submodule of C.

In this case, each C_α can also be a simple R-submodule by the well-known correspondence between the submodules of $_RC$ and those of $_{R/J(R)}C$.

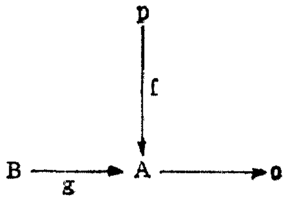
This implies that $_RC$ is the direct sum of simple R-submodules C_α of C.

Thus the Lemma 3 is established.

Ⅲ. Main Results

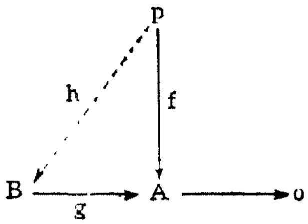
A left R-module P is projective in the following sense: Let g be a homomorphism of some module B onto A, then any homomorphism f : $P \longrightarrow A$ can be lifted to a homomorphism h : $P \longrightarrow B$ such that $g\cdot h=f$.

In the language of diagram, this means that every diagram



(그림 1)

in which the row is exact, can be embedded in a commutative diagram



(그림 2)

We recall that an exact sequence

(1) $\cdots\cdots\longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow A \longrightarrow 0$

is a projective resolution of A if each P_i, $(i=0, 1, 2, 3, \cdots)$, is a projective R-module. And $\mathrm{Ext}^n{}_R(A, C)$ is defined by the nth homology group $\mathrm{Ker}\ d'_{n+1}/\mathrm{Im}\ d_n'$ in the above diagram, where d_n' is as follows;

(2) $\mathrm{Hom}_R(P_0, C) \xrightarrow{d_1'} \mathrm{Hom}_R(P_1, C) \xrightarrow{d_2'} \mathrm{Hom}_R(P_2, C) \xrightarrow{d_3'} \cdots\cdots \xrightarrow{d'_{n-1}} \mathrm{Hom}_R(P_{n-1}, C) \xrightarrow{d_n'} \mathrm{Hom}_R(P_n, C) \xrightarrow{d'_{n+1}} \cdots\cdots.$

In this case, we may use the notation $\mathrm{Ext}^n(A, C)$ instead of $\mathrm{Ext}^n{}_R(A, C)$ without ambiguity of the base ring R.

Now, we state the main result.

Theoren 1

Let A be an R-module. If $\mathrm{Ext}^1{}_R(A, S)=0$ for some simple R-module S then $\mathrm{Ext}^1{}_R(A, C)=0$ for any R-module C such that $J(R)C=0$, where $J(R)$ is the Jacobson radical of the base ring R.

Proof. Let the following diagram be a projective resolution of A.

$\cdots\cdots\longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow A \longrightarrow 0.$

Then we have an exact sequence

(3) $0 \longrightarrow \mathrm{Hom}(P_0, C) \xrightarrow{d_1'} \mathrm{Hom}(P_1, C) \xrightarrow{d_2'} \mathrm{Hom}(P_2, C) \xrightarrow{d_3'} \cdots\cdots.$

Now that $J(R)C=0$, C can be represented as a direct sum of simple R-modules by Lemma 3. Let C denote a direct sum of some simple R-modules;

$$C = \oplus\sum_{i=1}^{n} S_i.$$

In this case, (3) can be represented as the following commutative diagram:

$$
\begin{array}{l}
(4)\ 0 \longrightarrow \mathrm{Hom}(P_0, \oplus\sum_{i=1}^{n} S_i) \xrightarrow{d_1'} \mathrm{Hom}(P_1, \oplus\sum_{i=1}^{n} S_i) \xrightarrow{d_2'} \mathrm{Hom}(P_2, \oplus\sum_{i=1}^{n} S_i) \longrightarrow \cdots\cdots \\
\qquad\qquad\qquad \wr\| \qquad\qquad\qquad\qquad \wr\| \qquad\qquad\qquad\qquad \wr\| \\
\quad 0 \longrightarrow \oplus\sum_{i=1}^{n} \mathrm{Hom}(P_0, S_i) \xrightarrow{d_1'} \oplus\sum_{i=1}^{n} \mathrm{Hom}(P_1, S_i) \xrightarrow{d_2'} \oplus\sum_{i=1}^{n} \mathrm{Hom}(P_2, S_i) \longrightarrow \cdots\cdots \\
\qquad\qquad\qquad \downarrow p_i^0 \qquad\qquad\qquad\qquad \downarrow p_i^1 \qquad\qquad\qquad\qquad \downarrow p_i^2 \\
\quad 0 \longrightarrow \mathrm{Hom}(P_0, S_i) \xrightarrow{d_{1i}''} \mathrm{Hom}(P_1, S_i) \xrightarrow{d_{2i}''} \mathrm{Hom}(P_2, S_i) \longrightarrow \cdots\cdots
\end{array}
$$

where p_i^0, p_i^1, p_i^2, $\cdots\cdots$ are i-th projections.

It can be easily verified that

$$\mathrm{Ext}^1(A, C) = \mathrm{Ker} d_2'/\mathrm{Im} d_1' = \oplus\sum_{i=1}^{n} \mathrm{Ker} d_{2i}''/\mathrm{Im} d_{1i}''.$$

By the hypothesis $\mathrm{Ext}^1(A, S_i) = \mathrm{Ker}\ d_{2i}''/\mathrm{Im}\ d_{1i}''=0$, we can conclude that $\mathrm{Ext}^1(A, C)=0$ for any R-module C such that $J(R)C=0$.

Therefore, the proof is complete.

It may be remarked that $J(R)C=J(C)$ is not always true, where $J(C)$ is the intersection of all maximal submodules of C.

But, assume that C is a projective R-module, then we shall also be able to come to the following conclusion, as a corollary of the Theorem 1.

Let A be an R-module. If $\mathrm{Ext}^1(A,S)=0$ for some simple R-module S then $\mathrm{Ext}^1(A,C)$ for any R-module C such that $J(C)=0$, where $J(C)$ is the intersection of all maximal submodules of C.

References

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