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On the boundedness of the set containing univalent functions

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〈Abstract〉

We denote a set of locally univalent functions defined on D by L.S. and S denote the subset of L.S. We consider X_3 as a norm topology defining a norm $\|f\|_3=\int_0^1 M(r:f)dr$. First, we explained properties of X_3 as a set, or a norm topological space.

Finally, we shall introduce the boundedness of S in X_3 on the compact open topology.

단엽함수를 포함하고 있는 집합의 유계성에 관하여

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〈요 약〉

복소수 평면상의 단위원 내부영역에서 국소단엽함수족들을 L.S.라 표시하고 그 부분집합을 S로 표시했다. 우리는 norm(노름)을 $\|f\|_3=\int_0^1 M(r:f)dr$로 정의하여 X_3을 norm 위상으로 생각했다.

처음에 X_3에 대한 집합 또는 norm 위상공간상에서 여러성질을 보았다. 결론으로 $L.S.$부분집합 S의 유계성을 compact개위상에서 증명한다.

I. Introduction

Let L.S. denote the family of locally schlicht holomorphic functions in $D=\{z:|z|<1\}$ and let S denote the family of schlicht functions in $L.S.=\{f:f$ analytic in $D,\ f(0)=0,\ f'(0)=1\}$.

In [1], [2], [3] and [4], the set L.S. can be given a real linear space structure with operations.

$$[f+g](z)=\int_0^z f'(\varphi)g'(\varphi)d\varphi \qquad (1\cdot1)$$
$$[\alpha f](z)=\int_0^z (f'(\varphi))^\alpha d\varphi$$

where f and g is in $L.S.$ and α is real.

And two diffe rent normed topological spaces, X_1 and X_2 were introduced, the next theorms were shown.

In this paper, we first study some results on X_3, and we shall introduce a theorem on X_3 and show the main theorem that S is bounded in X_3 on the compact open topology.

II. Some results on X and topological properties

In [1], [2], [3] and [4], if F is a linear-invariant family, the order of F is defined as the real number

$$\alpha=\sup\left\{\left|\frac{f''(0)}{2}\right|:f\in F\right\} \qquad (2\cdot1)$$

[4] is shown that $\alpha \geqslant 1$. And if

$$t(z,f) \equiv \left| -\bar{z} + \frac{(1-|z|^2)}{2} \frac{f''(z)}{f'(z)} \right| \quad (2\cdot 2)$$

then $\alpha = \text{order } F = \sup_{f \in F} \sup_{|z|<1} t(z,f)$.

Let $\mathscr{U}_\alpha$ denote the union of all linear-invariant family of oder $\leqslant \alpha$, that is,

$$\mathscr{U}_\alpha = \{f \in L.S. : \sup t(z,f) \leqq \alpha, \ |z|<1\} \quad (2\cdot 3)$$

Then we can see easily that for each α, $\mathscr{U}_\alpha$ is also a linear-invariant of order α. For any function $f(z)$ in X and any $0<\rho\leqslant 1$, we define $f_\rho(z) = f(\rho z)/\rho$.

Propsition 2.1 : If $f(z)$ is in $\mathscr{U}_\alpha$, then $f_\rho(z)$, $(0<\rho\leqslant 1)$ is also in $\mathscr{U}_\alpha$.

Proof. D.M. Campbell, Locally univalent functions with locally univalent derivative. Trans. Amer. Math. Soc. 162(1971), 395—409. results and $\sup_{|z|<1} t(z,f) \leqq \sup_{|z|<1} t(z,f)$ gives the proof.

Pommerenke [4] showed that if f is in $\mathscr{U}_\alpha$, there is a constant $K=K(\alpha)$ such that

$$|\log f'(z)| \leqq K \log \frac{1}{1-|z|} (z \in D) \quad (2.4)$$

Let X be the union of $\mathscr{U}_\alpha$, *i.e.*, $X = \bigcup_{\alpha \geq 1} \mathscr{U}_\alpha$ (2.5)

then X is a proper subset of $L.S.$ and is a real linear subspace of $L.S.$ with the induced operation (1.1). The above result is shown in [1], [2], [3] and [4], they introduced two different norm topology from a norm topolgy in this paper. As a result of that, they found several topological properties on them.

Ⅲ. On the results of X_3 with another topology

We shall introduce a different norm from that of X_1 and X_2 to equip X_3 with a normed topological space. If we define

$$M(\mathrm{r} : f) = \frac{1}{2\pi} \int_0^{2\pi} |\log|f'(\mathrm{r}e^{i\theta})|| d\theta \quad (3\cdot 1)$$

then we have for f & g in X_3 and α a real number

$$M(r : [f+g]) \leqq M(r : f) + M(r : g)$$
$$M(r : [\alpha f]) = |\alpha| M(r : f) \quad (3.2)$$

We use (3.1) &(3.2) to equip X_3 with a norm topology by defining the norm

$$\|f\|_3 = \int_0^1 M(r : f) dr, \text{ where } f \text{ is in } X_3 \quad (3.3)$$

We have written X_3 as the above normed topology.

Papers [1]&[3] let us obtain that the convergence in X_1 or X_2 implies the convergence in the compacta, as well as in X_3-topology. (shown in my the graduate degree paper). That is, convergence in X_3 implies convergence in the compact open topology. From this, we can see that X_3-topology is not weaker than the compact open topology. The partial converse of the above is shown by the following.

Proposition 3.1: If a sequence $\{f_n\}$ in $\mathscr{U}_\alpha$ converges to f in $\mathscr{U}_\alpha$ in compact open topology, then $\{f_n\}$ converges to f in the X_3 topology.

Proof (2.4) implies that for $|z|=r$ $(0<r<1)$,

$$|\log|f'_n(re^{i\theta})|| \leqq K \log \frac{1}{1-r},$$

where K is a constant, if we choose a integral both sides,

$$M(r : f) \leqq K \log \frac{1}{1-r}$$

then we can take $r_1 (0<r_1<r<1)$ such that

$$K \int_{r_1}^1 \log \frac{1}{1-r} dr < \frac{\varepsilon}{4}$$

and by the assumption, there exists an integer N such that for all $n \geqslant N$,

$$|\log|f'_n(z)| - \log|f'(z)|| < \frac{\varepsilon}{2} \quad (|z| \leqq r_1)$$

hence $M(r : [f_n - f]) < \frac{\varepsilon}{2}$ $(r \leqq r_1)$

Thus for such r_1 & N, if $n \geqslant N$ then

$$\|f_n - f\|_3 = \int_0^{r_1} M(r; [f_n - f]) \, dr + \int_{r_1}^1 M(r; [f_n - f]) dr < \frac{\varepsilon}{2} + \int_{r_1}^1 M(r : f_n) dr + \int_{r_1}^1 M(r : f) dr$$

The proof completes.

In order to prove the main theorem in this paper, we need the followings (Lemma and Definition).

Let $\mathscr{F}$ be the family of sets of holomorphic (or continuous) mappings on $D=\{z: |z|<1\}$

Definition 3.2: The subset F in $\mathscr{F}$ is said bounded if and only if for each compact set K contained in D,

$$\sup_{K}\{\|f\| : f\in F\}<\infty$$

That is, the definition tells that the functions in F are uniformly bounded on each compact subset.

Lemma 3.3: For F in $\mathscr{F}$, F is bounded iff each sequence in F has a convergent subsequence.

Proof. It follows from the problem 2[5. p. 168].

Theorem 3.4: The subst S of $L.S.$ (defined in 1. Introduction) is bounded in X_3 on the compact open topology.

Proof. As a result of Lemma 3.3, it's sufficient to show that each sequence in S has a convergent subsequence. Let f be in S, and $\{\rho_n\}$ be a sequence of positive number increasing to 1. For each $n=1, 2, 3, \cdots\cdots$,

$f_n(z)=\frac{f(\rho_n z)}{\rho_n}$ is analytic in $|z|<\frac{1}{\rho_n}$, and $f'_n(z)\neq 0$ on $\bar{D}$. Since S is $\mathscr{U}_\alpha$, by the *proposition* 2.1 $f_n(z)$ is contained in S. And from *proposition* 3.1, the sequence $\{\log|f_n'(z)|\}$ converges to $\log|f'(z)|$ uniformly, that is, $\{f_n(z)\}\longrightarrow f(z)\in S$ in X_3 on the compact open topology. Since uniformly convergent sequence has a convergent subsequence, by Lemma 3.3 the proof completes.

References

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