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On the Decoupling of Multivariable Systems by State Feedback

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⟨Abstract⟩

This paper treats the decoupling problem, which was considered by Falb and Wolovich[1] and Gilbert[2], more explicitly and rigorously in mathematical standpoint than they had taken and completes the proofs which they had omitted in the development of the theory. In particular, we show that the same results can be obtained without introducing the notion such as F-invariance or Integrator decoupled systems which was originally proposed by Gilbert[2].

상태 궤환에 의한 多變數系統의 個別制御에 關하여

金 容 人
應用數學科
(1980. 6. 10 접수)

⟨要 約⟩

본 논문에서는 Falb and Wolovich[1]와 Gilbert[2]가 제시했던 decoupling문제에 있어서 그들이 취했던 논리전개방법을 수학적으로 보다 간명하고 엄밀하게 취급했으며 또 그들이 생략했던 증명문제들을 완전히 하여 놓았다. 특히 Gilbert[2]가 도입했던 F-invariance나 Integrator decoupled system과 같은 개념을 고려하지 않아도 꼭같은 결론을 얻을 수 있음을 보였다.

I. Introduction

Consider the linear dynamical system with input u, output y and state x:

$$(1.1)\qquad \dot{x}=Ax+Bu$$
$$y=Cx$$

where

$u(t)=m\times1$ input control vector

$y(t)=m\times1$ output vector

$x(t)=n\times1$ state vector

and A, B, and C are constant matrices of size $n\times n$, $n\times m$ and $m\times n$, respectively.

Often one is interested in applying feedback control in order to implement certain control objectives. In conjunction with this approach, it is often of interest to know whether or not it is possible to have inputs control outputs independently, i.e. a single input influences a single output. This is so-called the problem of decoupling.

We denote the m-input, m-output, n-th order system (1.1) briefly by the triple $S=\{A,B,C\}$

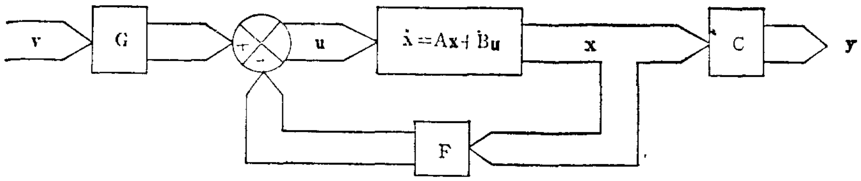
Now we consider the state feedback control law of the form

$$(1.2)\qquad u(t)=Fx(t)+Gv(t)$$

where F and G are constant matrices of size $m\times n$ and $m\times m$ respectively and $v(t)$ is the new $m\times1$ input vector into the closed-loop system (Fig.1)

Taking the Laplace transform of both sides of (1.1) with zero initial condition ($i.e.$ $x(0)=0$), we obtain

$$X(s)=(sI_n-A)^{-1}BU(s)$$



(Fig. 1. Multivariable state feedback system)

$$Y(s)=C(sI_n-A)^{-1}BU(s)$$

where s is a Laplace transform variable and I_n is the $n\times n$ identity matrix and $X(s)$, $U(s)$, $Y(s)$ are Laplace transforms of $x(t)$, $u(t)$, $y(t)$ respectively.

Thus we obtain the transfer function of the system (1.1):

$$(1.3)\qquad H(s)=\frac{Y(s)}{U(s)}=C(sI_n-A)^{-1}B$$

If we represent the control law (1.2) by the pair $\{F,G\}$ and the system (1.1) with the control law $\{F,G\}$ by $S(F,G)$, then we can obtain easily the following simple results.

Lemma 1:

$$S(F,G)=\{A+BF,BG,C\}\qquad(1.4)$$

$$H(s,F,G)=C(sI_n-A-BF)BG.\qquad(1.5)$$

where $H(s,F,G)$ is the transfer function of the system $S(F,G)$.

(proof) The system $S=\{A,B,C\}$ was defined by the equations.

$\dot{x}=Ax+Bu$ (state equation)

$y=Cx$ (output equation)

If we substitute the control law $u=Fx+Gv$ in to the above state equation, we obtain

$$\dot{x}=Ax+B(Fx+Gv)=(A+BF)x+BGv$$

$$y=Cx$$

$$\therefore\ S(F,G)=\{A+BF,BG,C\}$$

The last part of the proof is obvious by replacing A by $A+BF$ and B by BG in the equation (1.3).

A common control objective is to decouple the closed-loop system $S(F.G)$ (Fig. 1) by making the transfer function $H(s,F,G)$ be diagonal and nonsingular, *i.e.* causing

$$Y(s)=H(s,F,G)V(s)$$

$$\begin{bmatrix}Y_1(s)\\ \vdots\\ Y_m(s)\end{bmatrix}=\begin{bmatrix}h_1(s,F,G) & 0 & 0\cdots\cdots 0\\ 0 & h_2(s,F,G) 0 & 0\cdots\cdots 0\\ & \cdots\cdots & \\ 0 & 0 & h_m(s,F,G)\end{bmatrix}\begin{bmatrix}V_1(s)\\ \vdots\\ V_m(s)\end{bmatrix}$$

or, taking the inverse Laplace trausform,

$$y_i(t)=\bar{h}_i(t,F,G)v_i(t)\quad(i=1,\cdots,m)$$

where $\bar{h}_i(t,\ F,\ G)=\mathscr{L}^{-1}\{h_i(s,\ F,\ G)\}$.

Now we must examine the properties of $H(s, F,G)$ in order to obtain the necessary and sufficient conditions for decoupling.

Lemma 2. The transfer function $H(s,F,G)$ of the system $S(F,G)$ can be expanded in the following form.

$$(1.6)\ H(s,F,G)=q(s,F)^{-1}\,(CBs^{n-1}+CR_1(F)Bs^{n-2}+\cdots+CR_{n-1}(F)B)G$$

where

$$(1.7)\begin{cases}q(s,F)=|sI_n-A-BF|=s^n-q_1(F)s^{n-1}\cdots\\ \quad q_n(F)\\ R_0(F)=I_n\\ R_1(F)=(A+BF)-q_1(F)I_n\\ R_2(F)=(A+BF)^2-q_1(F)(A+BF)-q_2(F)I_n\\ R_{n-1}(F)=(A+BF)^{n-1}-q_1(F)(A+BF)^{n-2}\\ -\cdots-q_{n-2}(F)(A+BF)-q_{n-1}(F)I_n\end{cases}$$

(proof) We first expand $(sI_n-A)^{-1}$ and then extend these results to $(sI_n-A-BF)^{-1}$.

Let $P=(sI_n-A)^{-1}$.

By premultiplying (sI_n-A) to both sides of this equation, we obtain

$$sP=AP+I_n$$

and by premultiplying (sI_n+A) to both sides of this equation, we have

$$s^2P=A^2P+A+sI_n$$

In a similar way, by repeating this process we obtain the following set of equations·

$$P=P$$
$$sP=AP+I_n$$
$$s^2P=A^2P+A+sI_n$$
$$(1.8)\quad s^3P=A^3P+A^2+sA+s^2I_n$$
$$\cdots\cdots$$
$$s^nP=A^nP+A^{n-1}+sA^{n-2}+\cdots+s^{n-2}A+s^{n-1}I_n$$

Let the characteristic equation of A be

(1.9) $|sI_n-A|=s^n-q_1s^{n-1}-\cdots-q_{n-1}s-q_n=0$

Then from the Cayley-Hamilton theorem,

(1.10) $A^n-q_1A^{n-1}-\cdots-q_{n-1}A-q_nI_n=0$

If we multiply $-q_n$, $-q_{n-1}$, $\cdots$, $-q_1$, 1, to both sides of the first, the second, $\cdots$, and the nth equations of (1.8) respectively and sum up them side by side, we obtain

$$\begin{aligned}&(s^n-q_1s^{n-1}-\cdots-q_{n-1}s-q_n)P\\&=(A^n-q_1A^{n-1}-\cdots-q_{n-1}A-q_nI_n)P\\&\quad+(A^{n-1}-q_1A^{n-2}-\cdots-q_{n-2}A-q_{n-1}I_n)\\&\quad+s(A^{n-2}-q_1A^{n-3}-\cdots-q_{n-3}A-q_{n-2}I_n)\\&\quad\cdots\cdots\\&\quad+s^{n-2}(A-q_1I)\\&\quad+s^{n-1}I_n\end{aligned}$$

By making use of (1.9) and (1.10), we have

$$|sI_n-A|P=s^{n-1}R_0+s^{n-2}R_1+\cdots+sR_{n-2}+R_{n-1}$$

where

$$R_0=I_n$$
$$R_1=A-q_1I_n$$
$$\cdots\cdots$$
$$R_{n-1}=A^{n-1}-q_1A^{n-2}-\cdots-q_{n-1}I_n$$

$$\therefore\ P=(sI_n-A)^{-1}=q(s)^{-1}(s^{n-1}R_0+s^{n-2}R_1+\cdots+sR_{n-2}+R_{n-1})$$

where $q(s)=|sI-A|=s^n-q_1s^{n-1}-\cdots-q_{n-1}s-q_n$.

Replacement of A by $(A+BF)$, $q(s)$ by $q(s, F)$, q_i by $q_i(F)$ and R_i by $R_i(F)$ in the above equation gives the results (1.6) and (1.7)

Remark: The original system $S=\{A, B, C\}$ can be considered as a special case of the system $S(F, G)$ if we let $F=0$ and $G=I_m$. Thus all the relevant quantities of S can be written as follows: $H(s)=H(s, O, I_m)$, $q(s)=q(s, 0)$, $q_i=q_i(0)$, $R_i=R_i(0)$ etc.

Ⅱ. Formulation the problem

Now we give the definition of decoupling which was proposed by Falb and Wolovich [1] and make the precise development of discussion for the problem.

Definition 1: The control law $\{F, G\}$ decouples the system $S(F, G)$ if the transfer function $H(s, F, G)$ of $S(F, G)$ is diagonal and nonsingular.

We denote the ith row of $H(s, F, G)$ by $H_i(s, F, G)$ and in connection with this we define the integer $d_i(F, G)$ and the $1\times m$ row matrix $D_i(F, G)$ as follows.

Definition 2:

$$d_i(F,G)=\begin{cases}\text{integer } j \text{ such that}\\ \lim_{s\to\infty}s^{j+1}H_i(s,F,G)=\text{finite}(\neq 0)\quad (H_i(s,F,G)\neq 0)\\ n-1\quad (H_i(s,F,G)=0)\end{cases}$$

$$D_i(F,G)=\begin{cases}\lim_{s\to\infty}s^{j+1}H_i(s,F,G) & \text{if } H_i(s,F,G)\neq 0\\ 0 & \text{if } H_i(s,F,G)=0.\end{cases}$$

Lemma 3. $d_i=\min\{j : C_iA^jB\neq 0,\ j=0, 1, \cdots, n-1\}$

$$D_i=C_iA^{d_i}B\quad (i=1, \cdots, m)$$

(proof) Let C_i be the ith row of C. Putting $F=0$, $G=I_m$ in (1.6) and (1.7), we have

$$\begin{aligned}H_i(s)&=q(s)^{-1}[C_iBs^{n-1}+C_i(A-q_1I_n)Bs^{n-2}\\&\quad+C_i(A^2-q_1A-q_2I_n)Bs^{n-3}+\cdots\\&\quad+C_i(A^{n-1}-q_1A^{n-2}-\cdots-q_{n-2}A-q_{n-1}I_n)B]\\&=q(s)^{-1}[C_iB(s^{n-1}-q_1s^{n-2}-\cdots-q_{n-1})+\\&\qquad C_iAB(s^{n-2}-q_1s^{n-3}-\cdots-q_{n-2})+\\&\qquad C_iA^2B(s^{n-3}-q_1s^{n-4}-\cdots-q_{n-3})+\\&\qquad\cdots\cdots\\&\qquad C_iA^{n-2}B(s-q_1)+\\&\qquad C_iA^{n-1}B]\end{aligned}$$

where $q(s)=s^n-q_1s^{n-1}-\cdots-q_{n-1}s-q_n$

By the definition of d_i,

$$\begin{aligned}d_i&=\text{the integer } j \text{ such that } \lim_{s\to\infty}s^{j+1}H_i(s)\\&=\text{finite}\neq 0\\&=\begin{cases}0 & \text{if } C_iB\neq 0\\ j & \text{if } C_iB=0\end{cases}\end{aligned}$$

where j is the largest integer from $\{1, \cdots, n-1\}$ such that $C_iA^kB=0$ for $k=0, 1, \cdots, j-1$.

$$\therefore\ d_i=\min\{j : C_iA^jB\neq 0,\ i=0, 1, \cdots, n-1\}$$

$$\begin{aligned}D_i&=\lim_{s\to\infty}s^{d_i+1}H_i(s)\\&=\lim_{s\to\infty}\frac{s^{d_i+1}}{s^n-q_1s^{n-1}\cdots-q_n}\cdot[C_iA^{d_i}B(s^{n-1-d_i}\\&\quad-q_1s^{n-2-d_i}-\cdots-q_{n-1-d_i})\end{aligned}$$

$$+C_iA^{d_i+1}B(s^{n-2-d_i}-\cdots-q_{n-2-d_i})$$
$$+\cdots+C_iA^{n-1}B]$$
$$=C_iA^{d_i}B\ (i=1,\cdots,m)$$

Lemma 4. For the system S and $S(F,G)$,

$D_i(F,G)=D_iG$, where $D_i=D_i(0,I_m)$

$d_i(F,G)=d_i$ for $|G|\neq 0$,

where $d_i=d_i(0,I_m)$ $(i=1,\cdots,m)$

(proof) By Lemma 3, $d_i=\min\{j : C_iA^jB\neq 0, j=0,1,\cdots,n-1\}$ *i.e.*, $C_iA^kB=0$ for $k=0,1,\cdots,d_i-1$

$C_iA^{d_i}B=D_i\neq 0$.

Then by simple algebraic manipulations, we have

$$C_i(A+BF)=C_iA$$
$$C_i(A+BF)^2=C_i(A+BF)(A+BF)=C_iA(A+BF)=C_iA^2$$
$$C_i(A+BF)^k=C_iA^k\ (k=0,1,\cdots,d_i)$$
$$C_i(A+BF)^k=C_iA^{d_i}(A+BF)^{k-d_i}(k=d_i+1,\cdots,n)$$
$$\therefore\ d_i(F,G)=\min\{j : C_i(A+BF)^jB\neq 0,\ j=0,1,\cdots n-1\}$$
$$=\min\{j : C_iA^jB\neq 0, j=0,1,\cdots,n-1\}$$
$$=d_i$$

From the above discussion, we can see that

$$C_iR_k(F)B=C_i[(A+BF)^k-q_1(F)(A+BF)^{k-1}\cdots-q_k(F)I_n]B$$
$$=\begin{cases}0\ (k=0,1,\cdots,d_i-1)\\ D_i(k=d_i)\end{cases}$$
$$\therefore\ D_i(F,G)=\lim_{s\to\infty} s^{d_i(F,G)+1}H_i(s,F,G)$$
$$=\lim_{s\to\infty} s^{d_i+1}q(s,F)^{-1}[C_iBs^{n-1}+C_iR_1(F)Bs^{n-2}+C_iR_{n-1}(F)B]G$$
$$=\lim_{s\to\infty}\frac{s^{d_i+1}}{s^n-q_1(F)s^{n-1}-\cdots-q_n(F)}$$
$$[C_iR^{d_i}(F)Bs^{n-1-d_i}+C_iR_{d_i+1}(F)Bs^{n-2-d_i}+\cdots+C_iR_{n-1}(F)B]\cdot G$$
$$=C_iR_{d_i}(F)B\cdot G$$
$$=D_iG$$

Remark: Gilbert[2] has introduced the notion of F-invariant. He called the properties of $S(F,G)$ which are not affected by changes in F by F-invariant. In this context, we can say that $d_i(F,G)$ and $D_i(F,G)$ are F-invariants of S.

Ⅲ. Main Results for Decoupling

Theorem 1: Let D be the $m\times m$ matrix given by

$$D=\begin{bmatrix}D_1\\ \vdots\\ D_m\end{bmatrix}=\begin{bmatrix}C_1A^{d_1}B\\ \vdots\\ C_mA^{d_m}B\end{bmatrix}$$

If $D=\Gamma=\mathrm{diag}(\gamma_1,\cdots,\gamma_m)$ where $\gamma_i\neq 0$ $(i=1,\cdots,m)$ and $C_iA^{d_i+1}=0$ $(i=1,\cdots,m)$, then the transfer function $H(s)$ of S is also diagonal and nonsingular, *i.e.*, $H(s)=\mathrm{diag}\,(h_1(s),\cdots,h_m(s))$, where $h_i(s)=\gamma_i\,s^{d_i-1}$.

(proof) Letting $F=0$, $G=I_m$ in (1.6) and (1.7), we have

$$H_i(s)=q(s)^{-1}[C_iB(s^{n-1}-q_1s^{n-2}-\cdots-q_{n-1})+$$
$$C_iAB(s^{n-2}-q_1s^{n-3}-\cdots q_{n-2})+$$
$$\cdots\cdots$$
$$C_iA^{n-2}B(s-q_1)+$$
$$C_iA^{n-1}B]$$

By Lemma 3 and the hypothesis of the Theorem 1, we have

$C_iA^{d_i}B=D_i=\gamma_iE_i$ where $E_i=ith$ row of I_m

$C_iA^kB=0$ for $k\neq d_i$

$$\therefore\ H_i(s)=q(s)^{-1}[C_iA^{d_i}B(s^{n-1-d_i}-q_1s^{n-2-d_i}-\cdots-q_{n-1-d_i})]$$
$$=q(s)^{-1}(s^{n-1-d_i}-q_1s^{n-2-d_i}-\cdots-q_{n-1-d_i})\gamma_iE_i$$

Now from the Cayley-Hamilton theorem,

$$A^n-q_1A^{n-1}-\cdots-q_{n-1}A-q_nI_n=0.$$

Premultiplying C_iA^j and postmultiplying B to both sides of this equation, we have

$$C_iA^{n+j}B-q_1C_iA^{n+j-1}B-\cdots-q_nC_iA^jB=0$$
$$(j=0,1,2,\cdots)$$

Taking $j=0,1,\cdots,d_i$ and noting that the coefficient of $C_iA^{d_i}B$ must vanish, we obtain

$$q_{n-d_i}=q_{n-d_i-1}=\cdots=q_n=0$$
$$\therefore\ q(s)=s^n-q_1s^{n-1}-\cdots-q_{n-d_i-1}s^{d_i+1}$$
$$\therefore H_i(s)=q(s)^{-1}(s^{n-1-d_i}-q_1s^{n-2-d_i}-q_{n-1-d_i})\gamma_iE_i$$
$$=q(s)^{-1}\cdot q(s)\cdot s^{-d_i-1}\gamma_iE_i$$
$$=s^{-d_i-1}\gamma_iE_i$$

Remark: By Theorem 1, if

$D=\mathrm{diag}(\gamma_1,\cdots,\gamma_m)$, $\gamma_i\neq 0$ and $C_iA^{d_i+1}=0$ $(i=1,\cdots,m)$

then from the transfer properties of $H(s)$ of S,

$$Y(s)=H(s)U(s)$$

or
$$\begin{bmatrix} Y_1(s) \\ \vdots \\ Y_m(s) \end{bmatrix} = \begin{bmatrix} r_1 s^{-d_1-1} & 0 & 0 & \cdots & 0 \\ 0 & r_2 s^{-d_2-1} & 0 & \cdots & 0 \\ & \cdots\cdots & & & \\ 0 & 0 & & 0 & r_m s^{-d_m-1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ \vdots \\ U_m(s) \end{bmatrix}$$

or, $Y_i(s)=r_i s^{-d_i-1}U_i(s)$

or, taking the inverse Laplace transform,

$$y_i(t)=r_i\iint\cdots\int u_i(t)(dt)^{d_i+1}$$

that is, the ith output is merely the (d_i+1)-fold integral of the ith input.

Gilbert[2] called such system the integrator decoupled system.

Definitoin 3. $S=\{A, B, C\}$ and $\bar{S}=\{\bar{A}, \bar{B}, \bar{C}\}$ are control low equivalent $(C. L. E)$

$\Longleftrightarrow \exists a$ 1—1 correspondence between $\{F,G\}$ and $\{\bar{F},\bar{G}\}$ such that, for this conespondence $H(\cdot, F,G)=\bar{H}(\cdot,\bar{F},\bar{G})$.

Theorem 2: (1) $S=\{A, B, C\}$

(2) $D=\begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix}=\begin{bmatrix} C_1A^{d_1}B \\ \vdots \\ C_mA^{d_m}B \end{bmatrix}$ is nonsingular

(3) $A^*=\begin{bmatrix} C_1A^{d_1+1} \\ \vdots \\ C_mA^{d_m+1} \end{bmatrix}$

$\Rightarrow$ (i) S and $\bar{S}=S(F,G)=S(-D^{-1}A^*,D^{-1})$ are $C.L.E.$

(ii) $\bar{d}_i=d_i$, $\bar{D}_i=E_i$ $(i=1, \cdots, m)$

(proof) (i) $\bar{S}=S(F,G)=S(-D^{-1}A^*,D^{-1})$, where $F=-D^{-1}A^*$, $G=D^{-1}$, $u=Fx+Gv$

$=\{A+BF,BG,C\}$

$=\{A-BD^{-1}A^*,BD^{-1},C\}$

$=\{\bar{A},\bar{B},\bar{C}\}$

where $\bar{A}=A-BD^{-1}A^*$, $\bar{B}=BD^{-1}$, $\bar{C}=C$.

We wish to choose $\bar{F}$ and $\bar{G}$ so that $\bar{S}(\bar{F},\bar{G})=S$

and so $H(s,F,G)=\bar{H}(s,\bar{F},\bar{G})$

where $H(s,F,G)=C(I_ns-A-BF)BG$

$\bar{H}(S,\bar{F},\bar{G})=\bar{C}(I_nS-\bar{A}-\bar{B}\bar{F})\bar{B}\bar{G}$

But $A+BF=A-BD^{-1}A^*$ and

$\bar{A}+\bar{B}\bar{F}=(A-BD^{-1}A^*)+(BD^{-1})\bar{F}$

Thus if we choose $F=0$ and let $\bar{F}=DF+A^*$ then $A+BF=\bar{A}+\bar{B}\bar{F}$.

Similarly, since $BG=BD^{-1}$ and $\bar{B}\bar{G}=(BD^{-1})\cdot\bar{G}$, if we select $\bar{G}=I_m$ and let $\bar{G}=DG$, then $BG=\bar{B}\bar{G}$.

Hence if we establish the 1—1 correspondence between $\{F,G\}$ and $\{\bar{F},\bar{G}\}$ such that $DF+A^*=\bar{F}$ and $DG=\bar{G}$, then $S=\{A,B,C\}$ and $\bar{S}=\{\bar{A},\bar{B},\bar{C}\}$ are $C.L.E.$

(ii) Since $d_i=\min\{j: C_iA^jB\neq 0,\ \ j=0,\ 1,\ n-1\}$ and $C_i(A+BF)^k=C_iA^k(k=0,1,\ \cdots,\ d_i)$, we have

$\bar{d}_i=\min\{j:\bar{C}_i\bar{A}^j\bar{B}\neq 0,\ j=0,1,\cdots,n-1\}$

$=\min\{j: C_i(A+BF)^jBG\neq 0, j=0,1,\cdots,n-1\}$

$=d_i$ $(i=1,\ \cdots,\ m)$ where $F=-D^{-1}A^*$, $G=D^{-1}$.

$\bar{D}_i=\bar{C}_i\bar{A}^{\bar{d}_i}\bar{B}=C_i(A+BF)^{d_i}BD^{-1}$

$=D_iD^{-1}=E_i$

$\bar{C}_i\bar{A}^{\bar{d}_i+1}=C_i(A+BF)^{d_i+1}$

$=C_iA^{d_i}(A+BF)$

$=C_iA^{d_i+1}+C_iA^{d_i}BF$

$=A_i^*+D_i(-D^{-1}A^*)$

$=A_i^*-A_i^*=0.$

Theorem 3: (Necessary and Sufficient conditions for decoupling)

$S(F,G)$ is decoupled$\Longleftrightarrow DG=\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_m)$ $(\lambda_i\neq 0)$

(proof) $(\Rightarrow)$: assume that $S(F,G)$ is decoupled, *i.e.* $H(s,F,G)$ is diagonal and nonsingular or $H_i(s,F,G)=h_i(s,F,G)E_i$, $h_i(s,F,G)\neq 0$.

By using the definition of $D_i(F,\ G)$ and Lemma 4., we have

$D_i(F,G)=D_iG=\lim_{s\to\infty} s^{d_i+1}H_i(s,F,G)$

$=\lim_{s\to\infty} s^{d_i+1}h_i(s,F,G)E_i$

$=\lambda_iE_i$

where $\lambda_i=\lim_{s\to\infty} s^{d_i+1}h_i(s,F,G)$

$\neq 0$ by the def. of d_i.

$\therefore$ $DG=\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_m)$, Λ nonsingular.

$(\Leftarrow)$: assume that $GD=\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_m)$, $\lambda_i\neq 0$, we need only to show that $H(s,F,G)$ is diagonal and nonsingular

Let $A^*=\begin{bmatrix} C_1A^{d_1+1} \\ \vdots \\ C_mA^{d_m+1} \end{bmatrix}$

Since $GD=\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_m)$, $\lambda_i\neq 0$, $D=G^{-1}\Lambda$

is nonsingular and so D^{-1} exists.

Then, with the control law $\{F,G\}=\{-D^{-1}A^*, D^{-1}\}$, $\bar{S}=S(F,G)=S(-D^{-1}A^*, D^{-1})$ are *C. L. E.* to S by the Theorem 2. Furthermore $\bar{D}_i(F,G)=E_i$ or $\bar{D}(F,G)=I_m$

Therefore, by Theorem 1, $H(s, F, G)$ is diagonal and nonsingular and has diagonal element $h_i(s,F,G)=s^{-d_i-1}$.

Remark: Since $\bar{D}(F,G)=I_m=DG$ by Lemma 4, $D=G^{-1}=$nonsingular. Hence the system S can be decoupled iff D is nonsingular.

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