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Minimum Cost Network Flow Problem with Fuzzy Arc Capacities

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〈Abstract〉

In this paper we deal with the minimal cost network flow problem with fuzzy arc capacitity constraints. When the arc capacities are fuzzy with linear L-R type membership function, using parametric programming procedure, we reduced it to the deterministic minimal cost network flow problem which can be solved by various typical network flow algorithms such as the out-of-kilter algorithm. Also a modified algorithm is presented.

화지 아크용량을 갖는 네트워크의 최소비용 흐름 문제

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〈요 약〉

본 논문에서는 화지 아크용량을 갖는 네트워크의 최소비용흐름 문제에 관해 다룬다. 아크용량이 L-R형태의 멤버쉽 함수를 갖는 화지집합으로 정의되는 네트워크의 최소비용흐름 문제를 파라메트릭 프로그래밍 절차를 이용하여 Out-of-Kilter와 같은 앨고리즘으로 풀 수 있는 전형적인 네트워크의 최소비용흐름 문제로 변환하였다.

1. Introduction

There were many theoretical development in the various decision making problems which are formulated to deterministic models such as a linear programming model or a stochastic model. But whether they used deterministic or stochastic model, it was not sufficient to represent the reality of the problem because in many real decision situations the decision maker is not able to specify exactly the objective fuction and the constraints in such a way as normally assumed in classical decision theory. Neither a purely deterministic nor a probabilistic approach can be applied, because some quantities in such situaition can only be expressed in linguistic terms such as "much greater", "near to", or "less than".

In these cases fuzzy set theory might be helpful. Zadeh [11] proposed the concept of fuzzy subsets as a way to handle such problems. The

Bellman and Zadeh's paper "Decision making in a fuzzy environment" (1970) [1] triggered considerable research in the area of decision theory and its applications. Using the Bellman and Zadeh's approach, many fuzzy mathematical programming related papers were written such-as fuzzy linear programming [12], fuzzy transportation problem [3], and fuzzy network maximal flow problem [14].

There exists fuzziness in the network problems in the real world. For an example the supply of plants and the demand of distribution centers may be very fuzzy in the production-distribution problem. There may exist instability in the amount of material to buy from suppliers. The production-distribution problem can be formulated as a network problem. The plants and the distribution centers are corresponding to the sources and the sinks in the network, respectively. Also, the supply and demand is corresponding to the capacity of arcs. Therefore the minimum cost network flow problem with fuzzy arc capacity is more realistic than the conventional minimum cost network flow problem.

In this paper we deal with the minimal cost network flow problem with fuzzy arc capacity constraints. When the arc capacities are fuzzy-with linear L-R type membership function, the parametric programming procedure [2] is employed to reduce it to the deterministic minimal cost network flow problem, which can be solved by various typical network flow algorithms such as the Out-of-Kilter algorithm.

2. Fuzzy set and its membership fucntion

The general framework proposed by Bellman and Zadeh for fuzzy decision problem is presented as follows. Let X be a set of possible alternatives. A fuzzy goal G is a fuzzy subset in the set of all alternatives X characterized by its membership function

$$G : X \to [0,1] \tag{2.1}$$

A fuzzy constraint C is a fuzzy subset in X characterized by its membership function

$$C : X \to [0,1] \tag{2.2}$$

The fuzzy decision D resulting from the fuzzy goal G and the fuzzy constraint C is the intersection of both, *i.e*,

$$D = G \cap C, \tag{2.3}$$

where the membership function of μ_D is defined by

$$\mu_D(x) = \mu_G(x) \cap \mu_C(x) = \min(\mu_G(x),\ \mu_C(x)) \tag{2.4}$$

for each $x \in X$.

The real fuzzy number $\tilde{n} = (a_n, b_n, c_n)$ with a triangular membership function ([13], see Dubois and Prade(1980)) is defined as the fuzzy subset of the real line with the membership function $\mu_n(x)$ such that

$$\mu_n(x) = \begin{cases} 0, & \text{for } x \leq a_n \\ (x-a_n)/(b_n-a_n), & \text{for } a_n \leq x \leq b_n \\ (c_n-x)/(c_n-b_n), & \text{for } b_n \leq x \leq c_n \\ 0, & \text{for } x \geq c_n. \end{cases} \tag{2.5}$$

We refer to (a_n, c_n) as the support of n; a_n is the lower value, b_n the modal value, c_n the upper value of the fuzzy subset n of number approximately equal to b_n.

Arithmetic operations on twe fuzzy numbers $m = (a_m, b_m, c_m)$ and $\tilde{n} = (a_n, b_n, c_n)$ are derived from Zadeh's extention principle[13]. Let μ_m and μ_n denote the respective membership functions. Then the membership function μ_{m+n} of the sum $\tilde{m}+\tilde{n}$ is defined by

$$\begin{aligned} \mu_{m+n}(z) &= \max_{z=x+y} \min(\mu_m(x), \mu_n(y)) \\ &= \max_{x} \min(\mu_m(x), \mu_n(z-x)). \end{aligned} \tag{2.6}$$

In order to find the typical shape of μ_{m+n} we consider a real value w such that $0 \leq w \leq 1$, and cordinates x_m and y_n with $x_m \in [a_m, b_m]$ and $y_n \in [a_n, b_n]$ such that

$$\mu_m(x_m) = \mu_n(x_n) = w$$

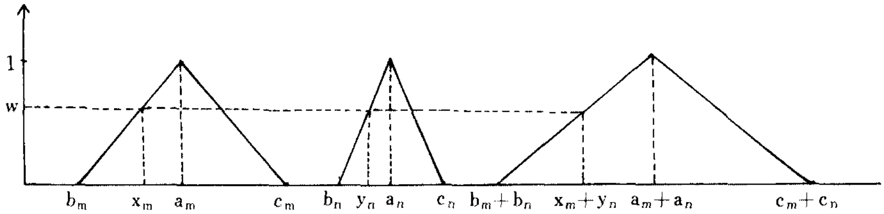

Fig. 1. Fuzzy numbers $\tilde{m}$ and $\tilde{n}$ with triangular membership functicn, and their fuzzy sum $\tilde{m}+\tilde{n}$.

Let $z^*=x_m+y_n$. Consider arbitrary coordinaes x and y such that $z^*=x+y$. If $x<x_m$, then $\mu_m(x)<w$, so that

$$\min\ (\mu_m(x),\ \mu_n(y))<w$$

We have the same result in the case of $x>x_m$, since $x>x_m$ implies $y<y_n$. It must accordingly be true that

$$\begin{aligned}\mu_{m+n}(x_m+y_n)&=\mu_{m+n}(z^*)\\&=\max_{z^*=x+y}\min(\mu_m(x),\ \mu_n(y))=w\\&=\mu_m(x_m)=\mu_n(y_n).\end{aligned}\tag{2.7}$$

This result holds for the increasing part of the membership function μ_{m+n} defined on the interval $[a_m+a_n,\ b_m+b_n]$. A similar result holds for the decreasing part on the interval $[a_m+a_n,\ c_m+c_n]$.

The membership function of $\tilde{m}+\tilde{n}$ can now be constructed; see Figure 1. Since it is triangular,

$$\tilde{m}+\tilde{n}=(a_m+a_n,\ b_m+b_n,\ c_m\ +c_n).$$

Scalar multiplication of fuzzy number, $\lambda\tilde{m}$, is also derived from Zadeh's extention principle. The membership function of is given by

$$\mu_{\lambda\cdot m}(x)=\mu_m(x/\lambda),\quad \text{for all } \lambda\in R-\{0\}\tag{2.8}$$

The membership function of $\lambda\tilde{m}$ can now be easily constructed; see Figure 2. It is also triangular, so

$$\lambda\cdot m==[\lambda a_m, \lambda b_m, \lambda c_m]$$

Obviously, we have the invariance in the parameters (a_m+a_n is lower value of the support, etc) and in the shape of the membership function (triangularity).

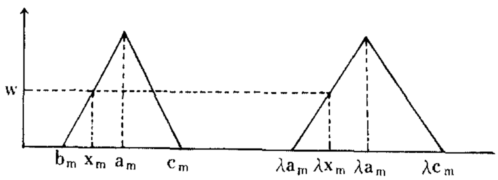

Fig. 2. Fuzzy numbers $\tilde{m}$ and their scalar multiplication

Prpposition 1. Consider the linear membership function defined by

$$z=\sum_i\sum_j\tilde{c}_{ij}x_{ij},\tag{2.9}$$

where each $\tilde{c}_{ij}$ is a fuzzy parameter with the mmbership function

$$\mu_c(c_{ij})=\begin{cases}1-\dfrac{|\alpha_{ij}-c_{ij}|}{p_{ij}}, & \text{if } \alpha_{ij}-p_{ij}\leq\tilde{c}_{ij}\leq \alpha_{ij}+p_{ij}\\ 0, & \text{otherwise}\end{cases}\tag{2.10}$$

The membership function of the linear function z is given by

$$\mu_z(z)=\begin{cases}\dfrac{1-\ |z-\sum_i\sum_j\alpha_{ij}x_{ij}|}{\sum_j\sum_i p_{ij}x_{ij}}, & x_{ij}\neq 0\\ 0, & \text{otherwise}\end{cases}\tag{2.11}$$

where $\mu_z(z)=0$ when $\sum_i\sum_j p_{ij}x_{ij}\leq|z-\sum_i\sum_j\alpha_{ij}x_{ij}|$

The proof is simple. Equation (2.9) is composed in terms of scalar multiplication of fuzzy number and their sums. Its membership function can be easily derived by operations of fuzzy numbers.

3. Minimum cost network flow with fuzzy arc capacities

Consider a directed network G, consisting of a finite set of nodes $N=1,2,\cdots,m\}$ and a set of directed arces $S=\{(i,j),\cdots,(k,1)\}$ joining pairs of nodes in N. A conventional minimal cost network flow problem may be stated as follows. Ship the available supply through the network to satisfy demand at the minimal cost. mathematically this problem becomes(where summation are taken over existing arcs)

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad (3.1)$$

$$\text{subject to} \quad \sum_{j=1}^{m} x_{ij}-\sum_{k=1}^{m} x_{ki}=0, \quad i=1,2,\cdots,m \qquad (3.2)$$

$$l_{ij}\le x_{ij}\le u_{ij}, \quad i,j=1,2,\cdots,m \qquad (3.3)$$

Constraint (3.3) represents the flow capacity of each arcs and constraint (3.2) is called the flow conservation equation. The flow conservation indicate that the flow may be neither created nor destoryed in the network. In (3.2), $\sum_{j=1}^{m} x_{ij}$ represents the total flow out of node i while $\sum_{k=1}^{m} x_{ki}$ indicates the total flow into node i.

The minium cost network flow problem with fuzzy capacities is represented as follows

$$\widetilde{M}\text{in} \quad \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij} \qquad (3.4)$$

$$\text{Subject to} \quad \sum_{j} x_{ij} - \sum_{k} x_{ki}=0, \quad i=1,2,\cdots,m \qquad (3.5)$$

$$\widetilde{l_{ij}}<x_{ij}<\widetilde{u_{ij}}, \quad i,j=1,2,\cdots,m \qquad (3.6)$$

where "$\sim$" represents the fuzziness in terms of tolerance limit. The constraints and goal are not crisp but fuzzy sets characterized by their membership functions.

We assume that the functions are of the following form. The membership function of the fuzzy goal is

$$\mu_G(z)=\begin{cases}1-t_0/p_0, & \text{if } z=z_0+t_0,\ t_0>0, t_0<p_0\\ 1, & \text{if } z<z_0 \\ 0, & \text{otherwise}\end{cases} \qquad (3.7)$$

where z_0 is the "aspiration level" for the objective function value and p_0 is the maximally acceptable violation of the level z_0.

In analogy the membership functions for the flow capacity constraints are defined as

$$\mu_L(l_{ij})=\begin{cases}1-t^l_{ij}/p^l_{ij}, & \text{if } l_{ij}=l^0_{ij}-t^l_{ij},\ t_{ij}\ge 0,\ t^l_{ij}\le p^l_{ij}\\ 1, & \text{if } l_{ij}\ge l^0_{ij} \\ 0, & \text{otherwise}\end{cases} \qquad (3.8)$$

and

$$\mu_U(u_{ij})=\begin{cases}1-t^u_{ij}/p^u_{ij}, & \text{if } u_{ij}=u^0_{ij}+t^u_{ij},\ t^u_{ij}\ge 0,\ t^u_{ij}\le p^u_{ij}\\ 1, & \text{if } u_{ij}\le u^0_{ij} \\ 0, & \text{otherwise}\end{cases} \qquad (3.9)$$

where l^0_{ij} and u^0_{ij} are "aspiration levels" of l_{ij} and u_{ij}, and respectively, and t^u_{ij} and t^l_{ij} are the maximally acceptable violation of level u^0_{ij} and l^0_{ij}, respectively. (See Figures 3, 4, and 5.)

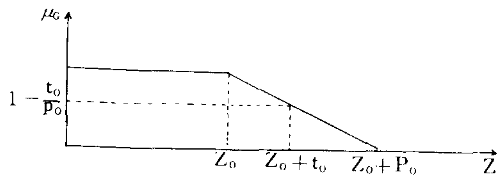

Fig. 3. (a) The membership function of the objective function

According to the Bellman and Zadeh's concept [1], a fuzzy decision is defined by a fuzzy set D with the membership function

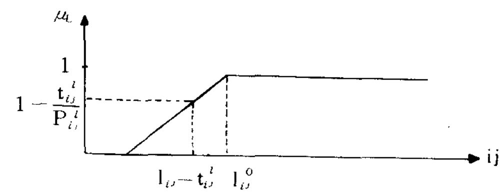

Fig. 3. (b) The membership function for the lower bound of capacities

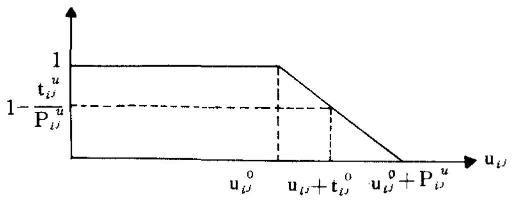

Fig. 3. (c) The membership function for the upper bound of capacities

$$\mu_D(x)=\mu_G(x)\cap\bigwedge_{(i,j)\in s}\mu_L(x)\cap\bigwedge_{(i,j)\in s}\mu_U(x) \quad (3.10)$$

where $\wedge$ is the minimum operator. The optimal decision x^* is the maximizing alternative such that

$$\mu_D(x^*)=\max_x \mu_D(x) \quad (3.11)$$

Chanas et. al [2] use the thechnique of parametric programming to solve the fuzzy linear programming problem. We show that equations (3.4)-(3.6) reduce to a deterministic minimal cost network flow problem.

Let us consider the fuzzy arc capacity constraint in the problem. Equation (3.6) can be represented as follows:

$$l^0_{ij}-\theta p^l_{ij} \leq x_{ij} \leq u^0_{ij}+\theta p^u_{ij},$$

where parameter $\theta(0<\theta<1)$ can be interpreted as the degree of the violation of an capacity constraint, it is easy to notice that for every admissible solution θx_{ij} of problem with a fixed parameter θ, the condition

$$\mu_U(\theta_{x_{ij}})>1-\theta \text{ and } \mu_L(\theta_{x_{ij}})>1-\theta \quad (3.13)$$
$$i,j=1,2,\cdots,m,$$

is valid. On the other hand for every feasible flow (if $p^u_{ij}>0$, $p^l_{ij}>0$ $i,j=1,2,\cdots,m$) there exists $(i,j)\in S$ such that

$$\mu_U(\theta_{x_{ij}})\wedge\mu_L(\theta_{x_{ij}})=1-\theta, \quad (3.14)$$

from which therefore the 'common' agree of satisfaction of the constraints can be given by

$$\bigwedge_{(i,j)\in s}\mu_U(\theta_{x_{ij}})\wedge(\bigwedge_{(i,j)\in s}\mu_L(\theta_{x_{ij}}))=1-\theta. \quad (3.15)$$

Hence the minimum cost network flow problem with fuzzy arc capacity can be reduced as follows dependent on parameter θ.

$$\text{Min} \sum_i \sum_j c_{ij}x_{ij} \quad (3.16)$$

$$s.t. \sum_j x_{ij}-\sum_k x_{ki}=0,\ i=1,2,\cdots,m \quad (3.17)$$

$$l_{ij}-\theta p^l_{ij}\leq x_{ij}\leq u^0_{ij}+\theta p^u_{ij},\ i,j=1,2,\cdots,m \quad (3.18)$$

Solving the above problem by the parametric programming thechnique [2], we obtain the set of solutions maximizing the objective function as analytically dependent on parameter θ, That is, for every θ we obtain a solution which satisfies the constraints with degree $1-\theta$ and simultaneously attains the goal in a possibly highest degree. The maximal value of the objective function of problem can now be presented asanalytically dependent on parameter θ and is a continuous, piece-wise linear and convex function in θ.

4. Algorithm

step 0: Set $\theta_0=0$ and $\theta_1=0.5$.
Update the arc capacities with θ_1.

step 1: Solve the problem by minimum cost network flow algorithm. If the solution is feasible, go to step 2; otherwise go to step 3.

step 2: If $|\mu_C-\mu_Z|\leq 0.005$, then stop. The optimal solution is obtained. Identify the optimal solution of current iteration Else if $\mu_C<\mu_Z$, go to step 4; oterwise go to step 5.

step 3: If $\mu_D>0$, then stop. Identify the optimal solution of previous iteration as the final optimal solution. If $\mu_D=0$, the problem is infeasible.

step 4: Set $\theta_0=\theta_1$ and $\theta_1=\theta_1-(\theta_1-\theta_0)/2$.
Update the arc capacities with θ_1. Go to step 1.

step 5: Set $\theta_0=\theta_1$ and $\theta_1=\theta_1+(\theta_0+\theta_1)/2$.
Update the arc capacities with θ_1. Go to step 1.

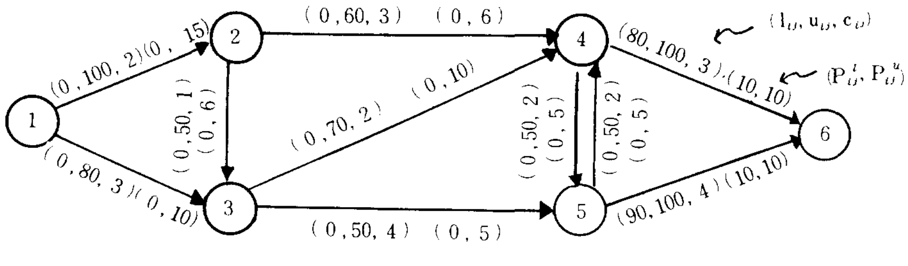

Fig. 4. The network structure of example problem and tolerance limit of arc capacities

5. Example

Consider the following the minimum cost network flow problem with 6 nodes and 10 arcs. Figure 4 shows the network structure and the arc capacity and its tolerance limit for each arc. The membership function of the objective function is given as follows:

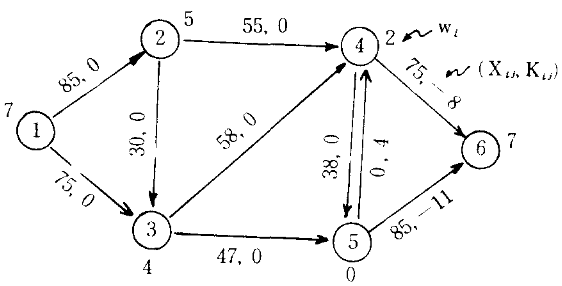

Fig.. 5(a) Network flow and kilter number for each arc of iteration 1 corresponding to $\theta=0.5$. ω_i: dual veriable for each node, k_{ij}: kilter state for each arc.

$$\mu_Z(z)=\begin{cases}1-(z-1500)/200, & 1500<z<1700,\\ 1, & z<1500,\\ 0, & \text{ptherwise.}\end{cases}$$

Suppose we use the out-of-kilter algorithm as the minimum cost algorithm. Then the Figure 5 show the flows for each itration.

The most satisfactory feasible flow is

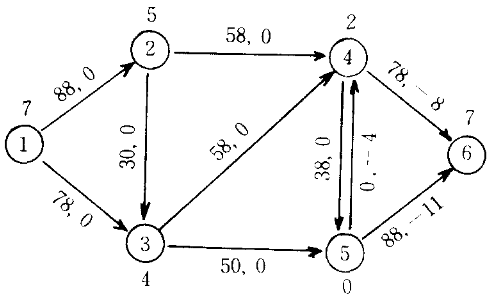

Fig. 5. (b) Network flow and kilter number for each arc of iteration 2 corresponding to $\theta=0.25$

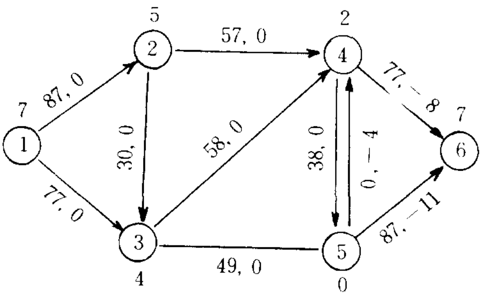

Fig. 5. (c) Network flow and kilter number for each arc of iteration 3 correspondiog to $\theta=0.375$

obtained as in Figure (5.d) and the obtained membership fucntion value of fuzzy decision μ_D is approximately 0.64. The value of μ_D corresponding to θ for each iteration is illustrated in Figure 6.

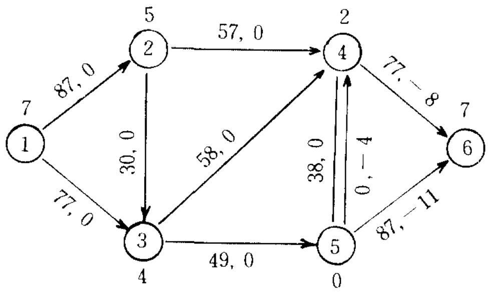

Fig. 5. (d) Network flow and kilter number for each arc of final iteration corresponding to $\theta = 0.64$

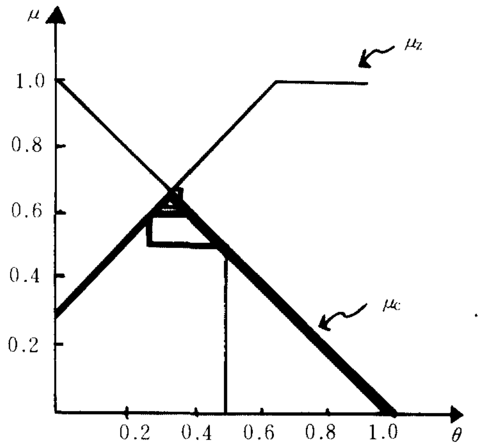

Fig. 6. The membership function of fuzzy decision μ_D.

References

1. R. E. Bellmanand L. A. Zadeh, Decision making in a fuzzy environment, Management Sci. 17, 1970, 141-164.
2. Stefan Chanas, The use of parametric programming in fuzzy linear programming, Fuzzy Sets and Systems 11, 1983, 243-251.
3. Stefan Chanas, Waldemar Kolodziejczyk, and Anna Machaj, A fuzzy approach to the transportaion problem, Fuzzy Sets and Systems 13, 1984, 211-221.
4. H. Hamacher, H. Leberling, and H. J. Zimmerman, Sensitivity analysis in fuzzy linear programming, Fuzzy sets and systems 1, 1978, 269-281.
5. F. A. Lootsma, Performance evaluation of nonlinear optimization methods via pairwise comparison and fuzzy numbers, Mathematical Programming 33, 1985, 93-114.
6. J. Llena, On fuzzy linear programming, European Journal of Operational Research 22, 1985, 216-223.
7. H. Tanaka and K. Asai, Fuzzy linear programming problems with fuzzy numbers, Fuzzy Sets and Systems 13, 1984, 1-10.
8. Roman Slowinsk, A multicriteria fuzzy linear programming method for water supply system development planning, Fuzzy Sets Systems 19, 1986, 217-237.
9. Jose L. Verdegay, A dual approach to solve the fuzzy linear programming problem, Fuzzy Sets and Systems 14, 1984, 131-141.
10. Ernest Czogala and Hans-Jurgen Zimmerman, Decision making in uncertain environments, European Journal of Operational Research 23, 1986, 202-212.
11. L. A. Zadeh, Fuzzy sets, Information anp control 8, 1965, 338-353.
12. H. J. Zimmerman, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1, 1978, 45-55.
13. Dider Dubois and Herri Prade, Fuzzy sets and Systems: Theory and Applications, Academic Press, 1980.
14. Stefan Chanas and Waldemar Kolodziejczyk, Integer Flows in Network with Fuzzy Capacity Constraints. Networks, 1986.