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ON BI-IDEAL AND QUASI-IDEAL OF Γ-SEMIGROUP

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ABSTRACT. In this paper we shall introduce the concept of a bi-ideal(quasi-ideal) in S where (S, Γ) is a Γ-semigroup and study some properties between S and M where M is the left operator semigroup of Γ-semigroup (S, Γ). We also give a characterization of regular Γ-semigroup

1. INTRODUCTION

In his pioneering paper [5], M.K.Sen and N.K.Saha introduced the concepts of a Γ-semigroups and many authors have studied the theory of Γ-semigroups([1,2,5]). D.K.Dutta and N.C. Adhikari [1] studied properties between Γ-semigroup and operator semigroup of Γ-semigroup. In this paper, bi-ideal and quasi-ideal of Γ-semigroup S are defined, and their relationships with corresponding structure in operator semigroup M of Γ-semigroup are established by similar methods.

Let S and Γ be nonempty sets. S is called a Γ-semigroup if there exists mappings

(1) $S \times \Gamma \times S \to S$, written as $(a, \alpha, b) \to a\alpha\beta$, and

(2) $\Gamma \times S \times \Gamma \to \Gamma$, written as $(\alpha, a, \beta) \to \alpha a\beta$,

such that $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$,for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. We call a Γ-semigroup S both sides if it futher satisfies the identities $\alpha a(\beta b\gamma) = (\alpha a\beta)b\gamma = \alpha(a\beta b)y\gamma$ for all $\alpha, \beta, \gamma \in \Gamma$ and $a, b \in S$.

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Throughout this paper we consider only both sided Γ-semigroup denoted by (S, Γ) and call it simply Γ-semigroup.

2. Preliminaries

We recall some notations and definitions ([1,5]). Let (S, Γ) be a Γ-semigroup. For $I, J \subseteq S$, we define $I\Gamma J = \{x\alpha\beta \mid x \in I, y \in J, \alpha \in \Gamma\}$

Let (S, Γ) be a Γ-semigroup. A nonempty subset I of S is called a left(right) ideal of a S if $S\Gamma I \subseteq I (I\Gamma S \subseteq I)$. If I is both a left and a right ideal then we say that I is two sided ideal or simply an ideal of S The ideals of Γ are defined analogously.

Let (S, Γ) be a Γ-semigroup. We define a relation ρ on $S \times \Gamma$ as follows: $(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Obviously ρ is an equivalence relation.

Let $M = \{[x, \alpha] : x \in S$ and $\alpha \in \Gamma\}$. and denote the equivalence class containing (x, α) by $[x, \alpha]$ We define a multiplication in M by $[x, \alpha][y, \beta] = [x\alpha y, \beta] = [x, \alpha y\beta]$. Then M forms a semigroup, and we call the left operator semigroup of the Γ-semigroup (S, Γ). Dually we can define the right operator semigroup $N = \{[\alpha, a] : \alpha \in \Gamma$ and $a \in S\}$ of the Γ-semigroup (S, Γ) where the multiplication in N is defined by $[\alpha, a][\beta, b] = [\alpha a\beta, b] = [\alpha, a\beta b]$.

The set S is a left M-set and a right N-set under the definitions $[a, \alpha]s = a\alpha s$ and $s[\beta, b] = s\beta b$. Futhermore $([a, \alpha]s)[\beta, b] = [a, \alpha](s[\beta, b])$. So S is (M, N)-biset. Similary Γ is a (N, M)-biset.

For $P \subseteq M(N)$ we define $P^+ = \{x \in S : [x, \alpha] \in P$ for all $\alpha \in \Gamma\}$, ${}^+P = \{\gamma \in$

$\Gamma : [x,\gamma] \in P$ for all $x \in S\}$, (resp. $P^* = \{x \in S : [\alpha, x] \in P$ for all $\alpha \in \Gamma\}$, ${}^*P = \{\gamma \in \Gamma : [\gamma, x] \in P$ for all $x \in S\}$.

Similary for $Q \subseteq S(\Gamma)$ we define $Q^{+'} = \{[x,\alpha] \in M : x\alpha s \in Q$ for all $s \in S\}$ and $Q^{*'} = \{[\alpha, x] \in N : s\alpha x \in Q$ for all $s \in S\}$ (resp. ${}^{+'}Q = \{[x,\alpha] \in M : \gamma x \alpha \in Q$ for all $\gamma \in \Gamma\}$ and ${}^{*'}Q = \{[\alpha, x] \in N : \alpha x \gamma \in Q$ for all $\gamma \in \Gamma\}$.) [1,5]

Definition 1.1 [5]. *Let (S,Γ) be a Γ-semigroup,M be the left operator semigroup and N be the right operator semigroup of (S,Γ). b We say that S has the left unity (right unity) if there exists an element $[e,\delta] \in M([\delta, e] \in N)$ such that $e\delta x = x$ $(x\delta e = x)$ for all $x \in S$. Similarly we can define the left unity and right unity of Γ.*

Propositin 1.2 [5]. *Let (S,Γ) be a Γ-semigroup.*

(1) *If $[e,\delta]$ is a left unity of S and a right unity of Γ then $[e,\delta]$ is an identity element of M.*

(2) *If $[\delta, e]$ is a left unity of Γ and a right unity of S then $[\delta, e]$ is an identity element of N.*

Definition 1.3 [5]. *Let (S,Γ) be a Γ-semigroup. (S,Γ) is said to be Γ-semigroup with unities if S has left unity and right unity which are also right unity and left unity of Γ respectively and vice-versa.*

2.Results

We first establish some relationships between bi-ideal of S and M.

Definition 2.1 [5]. *Let (S,Γ) be a Γ-semigroup. Let B be a nonempty subset of*

a Γ-semigroup M. The subset B is called a bi-ideal of M if $B\Gamma M\Gamma B \subseteq B$ Every one-sided ideal is bi-ideal.

Proposition 2.2. *Let (S,Γ) be a Γ-semigroup. If A is a bi-ideal of M, then A^+ is a bi-ideal of S. If S has a left unity and B is a bi-ideal of S ,$B^{+'}$ is a bi-ideal of M.*

Proof. Since $A \subseteq M$, we have that $A^+ \subseteq S$. Let $a, b \in A^+$, $\gamma, \mu, \nu \in \Gamma$,$s \in S$. Then $[a,\gamma], [b,\nu] \in A$. Since A is a bi-ideal of M, $[a,\gamma][s,\mu][b,\nu] \in A$ and so $[a\gamma s\mu b, \nu] \in A$. Hence $a\gamma s\mu b \in A^+$, whence $A^+\Gamma S\Gamma A^+ \subseteq A^+$. Therefore A^+ is a bi-ideal of S.

Assume that S has a left unity $[e,\delta]$ and B is bi-ideal of S. Let $[a,\alpha], [b,\beta] \in B^{+'}$, $[x,\epsilon] \in M$ and $y \in S$. By Definition of $B^{+'}$, we have $a\alpha e \in B$ and $b\beta y \in B$. Since $\delta x\epsilon \in \Gamma$ and B is a bi-ideal, $(a\alpha e)(\delta x\epsilon)(b\beta y) \in B\Gamma B \subseteq B$, and so $[a,\alpha][e\delta x,\epsilon][b,\beta]y \in B$ Hence $B^{+'}MB^{+'} \subseteq B^{+'}$. Therefore $B^{+'}$ is a bi-ideal of M.

Lemma 2.3.

(1) *Let $A, B \subseteq M$. Then $A^+\Gamma B^+ \subseteq (AB)^+$*

(2) *Let $A \subseteq S$. Then $MA^{+'} \subseteq (S\Gamma A)^{+'}$ and $A^{+'}M \subseteq (A\Gamma S)^{+'}$*

Proof. (a) Let $a \in A^+$,$b \in B^+$, $\gamma.\mu \in \Gamma$. Then $[a\mu b, \gamma] = [a,\mu][b,\gamma] \in AB$. Hence $a\mu b \in (AB)^+$ and so $A^+\Gamma B^+ \subseteq (AB)^+$

(b) Let $[a,\alpha] \in A^{+'}$, $[b,\beta] \in M$, $s \in S$ Then $a\alpha s \in A$. Whence $[b,\beta][a\alpha]s = [b,\beta]a\alpha s = b\beta(a\alpha s) \in S\Gamma A$. Hence $[b,\beta][a,\alpha] \in (S\Gamma A)^{+'}$ and so $[b,\beta][a,\alpha] \in (S\Gamma A)^{+'}$ Similary , $A^{+'}M \subseteq (A\Gamma S)^{+'}$

Definition 2.4 [5]. *Let (S,Γ) be a Γ-semigroup. Let Q be a nonempty subset of a Γ-semigroup M. The subset Q is called a quasi-ideal of M if $Q\Gamma M \cap M\Gamma Q \subseteq Q$.*

Every one-sided ideal is a quasi-ideal and every ideal is a quasi-ideal.

Proposition 2.5.

(1) *If A is a quasi-ideal of M, then A^+ is a quasi-ideal of S.*

(2) *If B is a quasi-ideal of S, then $B^{+'}$ is a quasi-ideal of M.*

Proof. (a) Clearly, $A^+ \subseteq S$ by $A \subseteq M$. Since $M^+ = S$, we have, from Lemma 2.3 (1) , that $A^+\Gamma S = A^+\Gamma M^+ \subset (AM)^+$. Similary, $S\Gamma A^+ \subseteq (MA)^+$, whence $(A^+\Gamma S) \cap (S\Gamma A^+) \subseteq (AM)^+ \cap (MA)^+ = ((AM) \cap (MA))^+ \subseteq A^+$, since A is a quasi-ideal of S. Hence A^+ is a quasi-ideal of S.

(b) $B^{+'} \subseteq S$. By Lemma 2.3 (2), we have that $B^{+'}M \subseteq (B\Gamma S)^{+'}$ and $MB^{+'} \subseteq (S\Gamma B)^{+'}$. Hence $(B^{+'}S) \cap (SB^{+'}) \subseteq (S\Gamma B)^{+'} \cap (S\Gamma B)^{+'} = ((B\Gamma S) \cap (S\Gamma B))^{+'} \subseteq B^{+'}$, since B is a quasi-ideal of S. Therefore $B^{+'}$ is a quasi-ideal of S

We now establish some relationships between semiprime quasi-ideals of S and M.

Definition 2.6. *Let (S, Γ) be a Γ-semigroup. Let Q be a nonempty subset of a Γ-semigroup S. A bi-ideal or quasi-ideal P is called semiprime ideal if $x \in S, x\Gamma S\Gamma x \subseteq P$ implies $x \in P$*

Proposition 2.7.

(1) *Let P be a semiprime quasi-ideal of M. Then P^+ is a semiprime quasi-ideal of S.*

(2) *Let Q be a semiprime quasi-ideal of S. Then $Q^{+'}$ is a semiprime quasi-ideal of M.*

Proof. (a) Let P be a semiprime quasi -ideal of M. Then P^+ is a quasi-ideal of S(Proposition 2.7 (1)). Let $a \in S - P^+$. Then there exists $\alpha \in \Gamma$ such that $[a, \alpha] \notin P$. Since P is semiprime, there exists $[c, \gamma] \in M$ such that $[a, \alpha][c, \gamma][a, \alpha] \notin P$. Then $[a\alpha c\gamma a, \alpha] \notin P$, whence $a\alpha c\gamma a \notin P^+$. Hence $a\Gamma S\Gamma a \subsetneq P^+$. So P^+ is semiprime.

(b) Let Q be a semiprime quasi-ideal of S. Then $Q^{+'}$ is a quasi-ideal of M(Proposition 2.7 (2)).) Let $[a, \alpha] \in M - Q^{+'}$. Then $a\alpha x \notin Q$ for some $x \in S$. Since Q is semiprime , there exists $\gamma, \mu \in \Gamma, s \in S$ such that $(a\alpha x)\gamma s\mu(a\alpha x) \notin Q$. Therefore $[a, \alpha][x\gamma s, \mu][a, \alpha]x \notin Q$. Thus $[a, \alpha][x\gamma s, \mu][a, \alpha] \in Q^{+'}$. So $[a, \alpha]M[a, \alpha] \subsetneq Q^{+'}$. Therefore $Q^{+'}$ is semiprime ideal in S.

Definition 2.8. *Let (S, Γ) be a Γ-semigroup. An element a of a Γ-semigroup M is called regular if $a \in a\Gamma M\Gamma a$. A Γ-semigroup M is called regular if every element of M is regular.*

Proposition 2.9. *A Γ-semigroup S is regular semigroup if and only if every bi-ideal in S is semiprime.*

Proof. Let S be a regular Γ-semigroup and B any bi-ideal of S. Suppose $x\Gamma S\Gamma x \subseteq B$ for $x \in S$. Since S is regular, $x \in x\Gamma S\Gamma x \subseteq B$. Therefore B is semiprime ideal. Conversely , assume that every bi-ideal of S is semiprime. Let $a \in S$ and consider $B = a\Gamma S\Gamma a$. Then B is bi-ideal of S. Hence $a\Gamma S\Gamma a$ is semiprime for any $a \in S$. Since $a\Gamma S\Gamma a$ is semiprime, we have $a \in a\Gamma S\Gamma a$. Therefore S is regular Γ-semigroup.

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