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Characterization of Radical in a Left Artinian Ring

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〈Abstract〉

It is shown that a left R-module C can be represented as a direct sum of simple R-submodules if and only if $NC=0$, where R is left Artinian, and N is the Jacobson radical of R. Furthermore, it is proved that if R is a left Artinian ring, N the radical of R, and $NC=0$, then $\mathrm{Hom}_R(A, C) \cong \mathrm{Hom}_{R/N}(A/NA, C)$, where $\mathrm{Hom}_R(A, C)$ is the set of all R-module homomorphisms of an R-module $_RA$ to $_RB$.

좌 Artin환에서의 라디칼의 특성

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〈요 약〉

본 논문에서는 R을 좌 Artin환, N을 Jacobson 라디칼 이라할때 좌 R-module C가 단순 R-submodule 의 직화로 표시되기 위한 필요하고도 충분한 조건은 $NC=0$임을 보인다.

그리고 더욱 나아가서 R-module $_RA$에서 R-module $_RB$로 가는 모든 R-module 준동형의 집합은 하나의 R-module이 될 수 있는데 이것이 R/N-module A/NA에서 R/N-module C로 가는 모든 R/N-module 준동형의 집합과 동형임을 밝힌다.

I. Introduction

In this note, some properties of the Jacobson radical N in a left Artinian ring R will be investigated.

Jacobson (1) showed that, in any left Artinian ring, the Jacobson radical is the unique maximal nilpotent two-sided ideal containing all nilpotent left ideals.

By using this fact, we can show that a left R-module C can be represented as a direct sum of simple R-submodules if and only if $NC=0$, where R is a left Artinian ring with identity.

Throughout this note, R represents a left Artinian ring and R-module means left R-module and $_RC$ denotes a left R-module C.

$\mathrm{Hom}_R(A, B)$ denotes the set of all R-module homomorphisms from an R-module $_RA$ to $_RB$. In this case, $\mathrm{Hom}_R(A, B)$ can also be an R-module in the natural way.

II. Some Necessary Lemmas

We recall that a ring R is called left Artinian if R has the minimum condition on left ideals, namely, the left R-module $_RR$ is left Artinian. A module $_RM$ is called semi-simple if the module $_RM$ can be represented as a direct sum of simple R-submodules of M. And a ring R is called semi-simple if the R-module R is semi-simple, that is, $_RR$ can be represented as a direct sum of left ideals which are simple when considered as R-submodules.

Before characterizing our main goal of this note, we need the following lemmas which can be found in (2).

Lemma 1

If R is a left Artinian ring, then the factor ring R/N is a semi-simple ring with minimum condition on left ideals, where N is the radical of R.

Lemma 2 (Structure Theorem of Artinian Ring)

Every non-zero R-module is the direct sum of simple R-submodules if and only if R is the direct sum of a finite number of simple left ideals;

$$R=\oplus\sum_{i=1}^{n}L_i$$

where each L_i is a simple left ideal. Also $L_i=Re_i$ where $\{e_i\}^n{}_{i=1}$ is a set of orthogonal idempotents such that

$$\sum_{i=1}^{n}e_i=1\epsilon R$$

Ⅲ. Characterization of the Radical

Now we are in a position to state our main goal and prove it.

Theorem 1

Let N be the radical of a left Artinian ring R and $_RC$ an left R-module. If $NC=\left\{\sum_{i=1}^{n}n_iC_i \,\middle|\, n_i\epsilon N, c_i\epsilon C\right\}=0$, then C is the direct sum of simple R-submodules.

Proof. Since R is left Artinian, the factor ring R/N is, by Lemma 1, a semi-simple ring which is left Artinian. In order to make C an R/N-module, we define an operation $\sigma: R/N\times C\to C$ with $\sigma(r+N, C)=rC$. Then σ is well defined by $NC=0$. And it can be easily verified that C is a left R/N-module. From the above mentioned fact, R/N is semi-simple left Artinian, and by Lemma 2, $_{R/N}C$ can be represented as a direct sum of simple R/N-submodules such that

$$C=\oplus\sum_{\alpha}C_\alpha$$

where C_α is a simple R/N-submodule of C. In this case, each C_α can be also a simple R-submodule by well-known correspondence between the submodules of $_RC$ and those of $_{R/N}C$. This implies that $_RC$ is the direct sum of simple R-submodule $_RC_\alpha$ of $_RC$. Thus the proof is completed.

The following theorem is a converse of Theorem 1.

Theorem 2

Let R be a left Artinian ring. If a non-zero R-module $_RC$ is simple, then $NC=0$.

Proof. The set NC is clearly an R-submodule of $_RC$. Now, since $_RC$ is simple, $NC=0$ or $NC=C$. Suppose $NC=C$, then $N^kC=C$ where k is any positive integer. Here N is a nilpotent ideal in R, that is, $N^n=0$ for some positive integer n. Of course, in this case, $C=N^nC=0$. This is a contradiction. Therefore $NC=0$, and the proof is completed. Moreover, if $_RC$ is semi-simple, then Theorem 2 can be also proved.

Corollary

If $_RC$ is semi-simple, then $NC=0$ under the same condition as in Theorem 2.

Proof. Suppose C can be represented as follows;

$$C=\oplus\sum_{\alpha}C_\alpha$$

where $_RC_\alpha$ is simple. Then, $NC_\alpha=0$ for all α by Theorem 2. From this, we know that

$$NC=N(\oplus\sum_{\alpha}C_\alpha)=\oplus\sum_{\alpha}NC_\alpha=0$$

Under the condition $NC=0$, we shall investigate the set $\mathrm{Hom}_R(A, C)$, where $_RA$ is any R-module. The following proposition can be shown straightfowardly.

Proposition

Let R be a left Artinian ring and N the radical of R. If $NC=0$, then

$$\mathrm{Hom}_R(A, C)\cong\mathrm{Hom}_{R/N}(A/NA, C).$$

Proof. First of all, A/NA is a left R/N-module because $N(A/NA)=0$.

Similarly, C is also an R/N-module. Now, let us define a mapping η from $\mathrm{Hom}_R(A, C)$ to

$\mathrm{Hom}_{R/N}$ $(A/NA, C)$ such that

$$\eta: \mathrm{Hom}_R(A, C) \longrightarrow \mathrm{Hom}_{R/N}(A/NA, C)$$
$$f \longrightarrow f^*$$
$$\text{with } f^*(a+NA)=f(a),\ a\in A.$$

Then we can easily prove that the mapping η is a R-module isomorphism. Of course, both $\mathrm{Hom}_R(A, C)$ and $\mathrm{Hom}_{R/N}(A/NA, C)$ can be left R-modules by a natural way. Thus we know that

$$\mathrm{Hom}_R(A, C) \cong \mathrm{Hom}_{R/N}(A/NA, C)$$

References

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