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Fixed Rings of Automorphisms on Fully Right Idempotent Rings.

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〈**Abstract**〉

A fixed ring R^G of a fully right idempotent ring is fully right idempotent where G is a finite group of automorphism and G is a bijection on R. Moreover, if R is strongly semi-prime ring, then R^G is strongly semi-prime ring and a finite direct sum of simple rings with identity.

Fully Right Idempotent 환위의 Automorphisms의 고정환에 대하여

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〈요 약〉

fully right idempotent 환의 고정환 R^G는 다시 fully right idempotent ring이고 만약 R이 strongly semiprime 환이면 R^G는 항등원을 가지는 simple환들의 유한개의 direct 곱으로 표시된다.

J. Osterburg showed that the fixed ring R^G of R is a direct sum of simple rings. We will prove that by using another method in this paper.

A ring R is said to be fully right idempotent if every right ideal of R is idempotent, or equivalently, if $a\in(aR)^2$ for every $a\in R$. Let G be a group which acts on R. The skew group ring $R*G$ is defined by $R*G=\{rg\mid g\in G, r\in R\}$ with addition given component wise and multiplication given as follows; if $r, s\in R$ and $g, h\in G$, then $(rg)(sh)=rs^g gh$. If $x=\Sigma r_g g$ is an element of $R*G$ then the support of x is the set Supp $(x)=\{g\in G\mid r^g\neq 0\}$

Let G be a finite group of automorphisms acting on the ring R. The fixed ring R^G of R is defined by $R^G=\{r\in R\mid r^g=r$ for every $r\in R\}$. For any integer n, we will say that n is a bijection on R if $nR=R$ and $nx=0$ implies $x=0$ for any $x\in R$. If n is a bijection, then for any $x\in R$ there exists a unique $y\in R$ such that $y=x/n$. For any $x\in R$, define the trace of x to be $tr(x)=\Sigma r^g$, Then clearly $tr(x)\in R^G$.

Lemma 1. Let G be group which acts on R. If R is a fully right idempotent ring with identity, then $R*G/I$ is a flat left R-module for every ideal I of $R*G$.

Proof. It is sufficient to prove that $a(R*G\cap I)\subset aI$ for every $a\in R$(1. Corollary 11.23.). By induction with respect to n, we shall show that if $a(r_1g_1+\cdots r_ng_n)\in I$, $r_i\in R$, $g_iG\subset G$, then $a(r_1g_1+\cdots+r_ng_n)\in aI$. Since R is fully right idempotent, $ar_1=ar_1\sum_{i=1}^{m} b_iar_1c_i$ with some b_i, $c_i\in R$, and therefore $ar_1g_1=ar_1\sum_{i=1}^{m} b_1ar_1c_ig_1 =ar_1\sum_{i=1}^{m} b(ar_1g_1)c_i^{g_1^{-1}}\in aI$, which proves the case $n=1$. Now, assume that $n>1$. As above, there exists a_i, $b_i\in R$ such that $ar_n=ar^n\sum_{i=1}^{m}$

$b_i a r_n c_i$. If we set $y = r_n \sum_{i=1}^{m} b_i a(r_1 g_1 + \cdots + r_n g_n) c_i^{g_n^{-1}} \in I$, we see that $v = a(r_1 g_1 + \cdots + r_n g_n - y) \in I$ and the cardinality of Supp (v) is less than n, by induction hypothesis, there exists then some $z \in I$ such that $v = az$. It follows therefore that $a(r_1 g_1 + \cdots + r_n g_n) = a(y+z) \in aI$.

By using this lemma we will prove the following theorem.

Theorem 2. Let R be a fully right idempotent ring with identity and G a finite group acts on R. If $|G|$ is a bijection on R, then $R*G$ is fully right idempotent.

Proof. Since $|G|$ is a bijection on R it suffices to show that S/I is a flat left S-module for each ideal I of S where S is $R*G$ (4). By lemma 1 S/I is a flat left R-module. Hence for any $a \in I$, there exists an R-homomorphism θ: $S \to I$ such that $\theta(ga) = ga$ for all $g \in G$ (1. Proposition 11.27.). As is easily verified, the map $\hat{\theta}$: $S \to I$ defined by $\hat{\theta}(s) = |G| \Sigma_{g \in G} g^{-1} \theta(gs)$ is an S-homomorphism with $\hat{\theta}(a) = a$. Hence S/I is flat again by (2. Proposition 11.27). Consequently S is fully right idempotent.

Cororally 3. Let R be a fully right idempotent ring with identity and G a finite group of automorphisms of R such that G is a bijection on R. Then the fixed subring R^G is fully right idempotent.

Proof. Since $R^G = e(R*G)e$ where $e = |G|^{-1} \Sigma_{g \in G} g$ (4. Lemma 1.2). It is obvious that R^G is fully right idempotent.

To prove our main theorem we now introduce right strongly semiprime ring and give some lemma. A ring R is called a right strongly semiprime ring provided if I is an ideal of R and is essential as a right ideal then there exists a finite subset F of X with $r(F)=0$. where $r(F) = \{r \in R \mid rf = 0$ for any $f \in F\}$. On the other hand we know that if R is a semiprime ring, then an ideal I of R is essential as a right ideal iff $r(I)=0$. Therefore we see that a ring R is a right strongly semiprime ring iff R is a semiprime ring and $r(I)=0$ implies $r(F)=0$ for some $F \subset I$.

Lemma 4. The following conditions are equivalent.

(1) R is a finite direct sum of simple rings with identity.

(2) R is a right strongly semiprime, fully right idempotent ring.

Proof. (1)⇒(2) It is clear that R is a right stronly semiprime ring. In order to see that R is fully right idempotent it suffices to show that every simple ring with identity is fully right idempotent. In fact, if S is a simple ring with identity and I is a right ideal of S, then $I^2 = (IS)I = I(SI) = IS = I$. (2)⇐(1) Let I be an arbitrary rideal of R, and choose an ideal K of R which is maximal with respect to the property that $I \cup K = 0$. We set $L = I \oplus K$. Since R is semiprime and $(L \cup r(L))^2 = 0$, $r(L)$ has to be 0 by the choice of K. Hence, there exists a finite subset F of L with $r(F)=0$. Since the ideals generated by F is a right s-unital ring, there exists an $e \in S$ such taht $xe = x$ for all $x \in F$ (5. Theorem 1) Since $a - ea \in r(F) = 0$ for all $a \in R$, e is a left identity of R. Now, let b be an arbitrary element of R, and choose an element f such taht $(be-b)f = be-b$. Since $(be-b)f = bef - bf = bf - bf = 0$, we obtain $be = b$, which means that e is the identity of R. Since e belongs to L, we readily obtain $R = L = I \oplus K$. We therefore have seen that R is a finite direct sum of simple rings with identity.

Using that lemma we can prove the following theorem.

Theorem 5. If R is a finite direct sum of simple rings with identity, then R^G is a finite direct sum of simple rings with identity.

Proof. Since R is a fully right idempotent and strongly right semiprime ring, R^G is fully right idempotent by Cororally 3. And if $r(I)=0$ where I is an essential ideal of R^G, then there exists some ideal J of R such that $r(J)=0$

where $I=R^G\cap J$. Since R is strongly semiprime there exists a finite subset F of R such that $r(F)=0$. $F\cap R^G\subset I$ is finite and $r(F\cap R^G)=0$. Thus R^G is strongly semiprime. Hence R^G is a finite direct sum of simple rings with identity.

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