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On the space with monotone normality operators

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〈Abstract〉

In this paper, we will introduce the relations between a stratifiable space and a monotonically normal space and show some properties of monotonically normal space.

Monotone normality operator를 갖는 공간에 관하여

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〈요 약〉

Monotonically normal space와 stratifiable space사이의 관계 및 monotonically normal space의 몇가지 성질을 연구함.

Ⅰ. Introduction

The property of monotone normality first appears, without name, in Lemma 2.1 of C. R Borges' paper "On stratifiable spaces" [1]. In [5], P. Zener named the property and announced results relating monotone normality to metrizability and stratifiability. In this paper, we will introduce the relations between a stratifiable space and a monotonically normal space and show some properties of monotonically normal space with more easy ways. The concept of monotone normality is used to give necessary condition for stratifiability of a T_1-s space to provide an easy proof of a metrization theorem.

Ⅱ. Definitions.

Throughout this paper all spaces are assumed to be at least T_1 and the set of natural numbers is denoted by the letter N.

Definition 2.1. A space X is stratifiable if to each closed set A of X one can assign a sequence $G_1(A)$, $G_2(A)$, ... of open subsets of X such that

(a) $A=\bigcap_{n=1}^{\infty} G_n(A)=\bigcap_{n=1}^{\infty} \overline{G_n(A)}$

(b) If $A\subset B$ are closed subsets of X then $G_n(A)\subset G_n(B)$ for each $n\in N$.

Definition 2.2. A space X is monotonically normal if there is a function D which assigns to each ordered pair (H, K) of disjoint closed subsets of X an open set $D(H, K)$ such that

(a) $H\subset D(H, K)\subset \overline{D(H, K)}\subset X-K$

(b) If (H', K') is a pair of disjoint closed sets having $H\subset H'$ and $K\supset K'$ then $D(H, K)\subset D(H', K')$.

The function D is called a monotone normality operator for X.

Definition 2.3. A space X is collectionwise normal if for each discrete collection $H=\{H_\alpha : \alpha \in \Lambda\}$ of closed subsets of X there is a disjoint collection $G=\{G_\alpha : \alpha \in \Lambda\}$ of open subsets of X with the property that $H_\alpha \subset G_\alpha$ for each $\alpha \in \Lambda$.

III. Properties of monotonically normal space

Theorem 3.1. Any stratifiable space is monotonically normal.

Proof. Suppose X is a stratifiable space. For each ordered pair (H,K) of disjoint closed subsets of X, let

$$D(H,K)=\bigcup_{n=1}^{\infty}[X-\overline{G_n(K)}-\overline{X-\overline{G_n(H)}}].$$

If $p \in H$, then there exists an $n \in N$ such that $p \notin \overline{G_n(K)}$ and hence $p \in X-\overline{G_n(K)}$. Now for each $n \in N$ $p \in G_n(H) \subset \overline{G_n(H)}$, hence $p \notin X-\overline{G_n(H)}$ and $p \notin X-G_n(H)$. Since $X-\overline{G_n(H)} \subset X-G_n(H)$, $p \notin \overline{X-\overline{G_n(H)}}$ for each n.

Thus $p \in D(H,K)$. Therefore, $H \subset D(H,K) \subset \overline{D(H,K)}$.

Let $p \in \overline{D(H,K)}$ and suppose $p \in K$, then $p \in \overline{D(K,H)}$. Let $q \in D(K,H) \cap D(H,K)$ and $k,n \in N$ such that $q \in X-\overline{G_n(K)}-\overline{X-\overline{G_n(H)}}$ and $q \in X-\overline{G_k(H)}-\overline{X-\overline{G_k(K)}}$. If $k<n$, then $G_n(H) \subset G_k(H)$ implies $q \in X-\overline{G_k(H)} \subset X-\overline{G_n(H)} \subset X-G_n(H)$. Thus $q \in \overline{X-\overline{G_n(H)}}$, we have a contradiction. Similary for $k>n$, hence we have $\overline{D(H,K)} \subset X-K$. Therefore, $H \subset D(H,K) \subset \overline{D(H,K)} \subset X-K$.

Finally let $p \in D(H,K)$, then there exists an $n \in N$ such that $p \in X-\overline{G_n(K)}-\overline{X-\overline{G_n(H)}}$. On the other hand, if $H \subset H'$ and $K \supset K'$, then $G_n(H) \subset G_n(H')$, $G_n(K) \supset G_n(K')$ for each $n \in N$. Thus $p \in X-\overline{G_n(K)}$ implies $p \in X-\overline{G_n(K')}$ for each $n \in N$. $X-G_n(H') \subset X-\overline{G_n(H)}$ implies $\overline{X-\overline{G_n(H')}} \subset \overline{X-G_n(H)}$. Since $p \notin \overline{X-\overline{G_n(H)}}$, $p \notin \overline{X-\overline{G_n(H')}}$. Hence $p \in D(H',K')$. Therefore, $D(H,K) \subset D(H',K')$.

Theorem 3.2. Any monotonically normal space is collectionwise normal.

Proof. Let X be a monotonically normal space and $F=\{F_\alpha : \alpha \in \Lambda\}$ a discrete collection of closed sets. Now well order by $\alpha_0, \alpha_1, \alpha_2, \cdots$.

For each $\alpha \neq \beta$, let $G_\alpha = D(F_\alpha, F_\beta)-\overline{D(F_\beta, F_\alpha)}$ and $G_\beta = D(F_\beta, F_\alpha)-\overline{D(F_\alpha, F_\beta)}$.

Then $G_\alpha \cap G_\beta = D(F_\alpha, F_\beta) \cap \overline{D(F_\beta, F_\alpha)}^c \cap D(F_\beta, F_\alpha) \cap \overline{D(F_\alpha, F_\beta)}^c \subset$

$D(F_\alpha, F_\beta) \cap D(F_\beta, F_\alpha)^c \cap D(F_\beta, F_\alpha) \cap D(F_\alpha, F_\beta)^c = \phi$.

Thus G_α and G_β are disjoint. By transfinite induction, we can get a collection $G_{\alpha_0}, G_{\alpha_1}, G_{\alpha_2}, \cdots$ of mutually disjoint open sets.

Suppose that there exists an $x \in F_\alpha$ such that $x \in \overline{D(F_\beta, F_\alpha)}$. Then $x \in X-F_\alpha$ and hence $x \notin F_\alpha$. Thus we have a contradiction. Hence $F_\alpha \subset G_\alpha = D(F_\alpha, F_\beta)-\overline{D(F_\beta, F_\alpha)}$. Therefore, X is collectionwise normal.

Theorem 3.3. Every subspace of a monotonically normal space is monotonically normal.

Proof. Let X be a monotonically normal space and A its subspace. Let D be a monotone normality operator such that for each disjoint closed subsets C and E $D(C,E) \cap D(E,C)=\phi$. For each disjoint closed sets H and K of A, let $D_A(H,K)=\bigcup_{x \in H}[D(\{x\},\overline{K}) \cap A]$. Clearly, $D_A(H,K)$ is open and $H \subset D_A(H,K) \subset \overline{D_A(H,K)}$. Let $p \in \overline{D_A(H,K)}$ and suppose $p \in K$. Then $p \in D_A(K,H) \subset A-H$. Let $q \in D_A(H,K) \cap D_A(K,H)$. Then $q \in D_A(H,K)$ and $q \in D_A(K,H)$. This implies that there exists an $x \in H$ such that $q \in D(\{x\},\overline{K}) \cap A$ and $y \in K$ such that $q \in D(\{y\},\overline{H}) \cap A$. Thus $q \in D(\{x\},\overline{K}) \cap D(\{y\},\overline{H})$. So we have a contradiction. Therefore, $H \subset D_A(H,K) \subset \overline{D_A(H,K)} \subset A-K$. If $H \subset H'$ and $K \supset K'$, then $D_A(H,K)=\bigcup_{x \in H}[D(\{x\},\overline{K}) \cap A] \subset \bigcup_{x \in H}[D(\{x\},\overline{K'}) \cap A] \subset \bigcup_{x \in H'}[D(\{x\},\overline{K'}) \cap A] = D_A(H',K')$. Hence $D_A(H,K) \subset D_A(H',K')$.

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