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The Relations Between The Tensor Space And The Symmetric Permutation Group

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〈Abstract〉

The relation between the tensor space which is induced on a vector space and the symmetric permutation group space is regorously investigated. The operations ♯ and ♮ which relate the two spaces are introduced. By these we obtain the close analogy between the two spaces.

텐서 공간과 대칭 치환군과의 관계

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〈요 약〉

벡터 공간으로 부터 유도되는 텐서 공간과 치환군의 공간과의 관계를 조사하였다.
새로운 연산자 ♯ 와 ♮ 을 도입하였고 이를 이용하여 두 공간의 긴밀한 연계성을 밝혔다.

Ⅰ. Introduction

The state of an r-identical particle system can be described by a tensor in a tensor space which is induced on a vector space of the one particle system.

The tensor space thus obtained is in general reducible; i. e. it has one or more nontrivial invariant subspaces in it.

The reduction of this tensor space into its irreducible components is usually made by the Clebsch-Gordan coefficients. (1)

Another approach is the use of the Young diagrams. (2,3)

In this paper we rigorously investigate the relations between the group space of symmetric permutation and the tensor space induced on a one particle vector space.

Ⅱ. Tensor space and the Symmetric transformations

We can define a transformation in the tensor space which is obtained from a vector space by the induction of a transformation in the vector space. (4,5)

This means a transformation in the vector space induces a transformation in the tensor space. The induced transformation is expressed as

$$G^{i_1\cdots i_r}=a_{i_1k_1}a_{i_2k_2}\cdots a_{i_rk_r}F^{k_1\cdots k_r}$$

for the transformation (a_{ij}) in the vector space

So this gives a representation of the transformations. This representation is in general reducible.

By a specific example, consider the case when $r=2$. In this case the symmetric tensors remain symmetric under these transformations. And the anti-symmetric tensors remain anti-symmetric under the transformations. Therefore these tensors form the invariant classes under these transtormations. Later we will investigate whether these subclasses are irreducible.

The coefficients of the above transformations have the following special properties; if we had the indexes $(i_1\cdots i_r)$ and $(k_1\cdots k_r)$ undergo the same symmetric permutation separately, then the coefficients in the whole do not change. We call these special transformations the symmetric transformations in the tensor space. The transformations having this property are writeen by

$$G^{i_1\cdots i_r}=C(i_1\cdots i_r\ ;\ k_1\cdots k_r)F^{k_1\cdots k_r}$$

and the C's can be expressed as the linear combinations of the certain induced transformation coefficients.

If we say the invariance and the irreducibility of a tensor space in the following these are under these symmetric tranformations considered above.

Ⅲ. Group space and the Representations of their algebras.

We now bring a new algebra into our considerations.[6,7]

An algebra is a set with the addition and the multiplication defined on it.

For every finite group we associate the real number field with this group.

This constitutes a new algebra. The multiplication on this algebra is defined in relation with the group multiplication law and the multication of real numbers. The addition is defined as in the case of the direct sum. The typical elements of this algebra is of the form

$$x=\sum_R x_R R.$$

where the summation is over all the elements of the finite group.

Two elements

$$x=\sum_R x_R R,\quad y=\sum_S y_s S$$

give under the addition

$$x+y=\sum_R (x_R+y_R)R$$

and give under the multiplication

$$xy=\sum_R(\sum_S x_S\, y_{S^{-1}R})R.$$

With these operations an algebra is well defined.

We can represent this algebra on the group space itself; for the element a we associate the left multiplication of this element with an arbitrary element x of the group algebra;

$$(a): x\longrightarrow x'=ax.$$

This certainly gives a representation.

In the following we will focus our attention into the reduction problem of this representation space.

An invariant subspace is the subspace p which satisfies

$$x'=ax\in p$$

where a is an arbitrary element of the group algebra and x is an element of the subspace p.

We can give a theorem for an invariant subspace.

Th. 1. If p is an invariant subspace, there exists an element $\hat{e}$ of the group algebra having the following two properties

(1) For any element $x\in c$, $xe\subset p$

(2) For any element $x\in p$, $xe=x$.

pf) Let $E_1, E_2, \cdots, E_g$ be the bases of the whole group space c and among these $E_1, E_2, \cdots, E_k$ form the bases of a linear subspace p.

The E_i's can be expressed as

$$E_i=\sum_{j=1}^{g} U_{ij}X_j$$

where X_j runs over all the elements of the finite group.

The inverse transformation $X_i \to E_j$ is obtained by the inverse matrix U^{-1}. Then define a map $x \dashrightarrow x' \in p$ by

$$x = x_1E_1 + \cdots + x_kE_k + \cdots + x_gE_g$$
$$\longrightarrow x' = x_1E_1 + \cdots + x_kE_k$$

In the (X_i) basis this map can be replaced by the matrix D where

$$D_{ij} = \sum_{k=1}^{k} U^{-1}{}_{ik} \text{ or } U_{kj}.$$

This D can be interpreted as the projection matrix. If the space p is invariant, the X_iE_1, $X_iE_2, \cdots X_iE_k$ form a new basis for the subspace p.

The transformation to this new basis is obtained by the matrix $U^{(i)}$;

$$E_k^{(i)} = \sum_j U_k^{(i)} X_j$$

and this $U^{(i)}$ is related to U by

$$U_{jk}^{(i)} = U_{jl}$$

where l is such that

$$X_l = X_i^{-1} X_k$$

The inverse of $U^{(i)}$ is related to U^{-1} by

$$[U^{(i)-1}]_{jk} = U^{-1}{}_{lk}$$

where l is such that $X_l = X_i^{-1} X_j$.

Using this basis the projection can be obtained by the matrix $D^{(i)}$;

$$D_{jk}^{(i)} = D_{lm}$$

where l, m is such that $X_l = X_i^{-1} X_j$, $X_m = X_i^{-1} X_k$.

Define the matrix E as

$$E = \frac{1}{g} \sum_{i=1}^{g} D^{(i)}$$

This matrix has the property $E_{jk} = E_{lm}$ if

$$X_j = X_i X_l, \quad X_k = X_i X_m$$

for an arbitrary element X_i.

Let $C_l = E_{ij}$ where l satisfies $X_l = X_j^{-1} X_i$ and $\hat{e} = c_l X_l$. Then the map $x \to x'$ can be obtained by this $\hat{e}$;

$$x' = x\hat{e}.$$

This establishes the theorem.

This $\hat{e}$ is unique for an invariant subspace as can be shown.

Therefore if $p = p_1 + p_2$, the $\hat{e}_1$ which generates p_1, plus $\hat{e}_2$ which generates p_2 can generate p. i.e. $\hat{e} = \hat{e}_1 + \hat{e}_2$.

And if p is reducible, it means a complete reducibility.

This can be shown. Let x be an element of an invariant subspace p and x_1 the component of x in p_1.

Then $x = x\hat{e}_1 + (x - x\hat{e}_1)$, and the set of $x - x\hat{e}_1$ is also invariant.

Ⅳ. The relations between the tensor space and the group space

We now define a operation on a tensor.

With a symmetric permutation S define a new tensor F' by

$$F'^{i_1 \cdots i_r} \equiv (SF)^{i_1 \cdots i_r} \equiv F^{j_1 \cdots j_r}$$

where $(j_1 \cdots j_r)$ transforms into $(i_1 \cdots i_r)$ by the permutation S; i.e.,

$$S = \begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix}$$

This definition satisfies the composition law of multiplication:

$$t(sF) = (ts)F.$$

And given a tensor, we can relate a set of elements of group space with this tensor.

An element $f[i_1 \cdots i_r]$ in group space is defined by

$$f[i_1 \cdots i_r] = \sum_R (RF)^{i_1 \cdots i_r} R$$

where the summation is over all the group elements.

With these operations we can relate the tensor space and the group space.

Let p be an arbitrary linear subspace of c. We can define the corresponding space in tensor space as follows; the corresponding space $\sharp p$ consists of all tensors which satisfies all the related elements $f[i_1 \cdots i_r]$ are in p for all combinations of $(i_1 \cdots i_r)$.

The $\sharp p$ thus obtained is necessarily invariant under the symmetric transformations of R^r.

And for a linear subspace β in R^r, we can associate a subspace $\natural\beta$ in group space.

Here we confine ourselves to the restricted case in which the dimension of the vector space R is larger than the number of identical particles r.

In this case $\natural\beta$ consists of all elements of the form $f[i_1\cdots i_r]$ related with the tensor $F\in\beta$. This $\natural\beta$ is also necessarily invariant under the left multiplication of group algebra.

Now if p is an invariant subspace, we can give the following theorems.

Before the formulation of the theorems we note a relation between the two spaces; let $F'=aF$. Then this is related to the corespondence

$$f'[i_1\cdots i_r]=f[i_1\cdots i_r]\hat{a}$$

for all $(i_1\cdots i_r)$where $\hat{a}$ is defined by $\hat{a}_s=a_s^{-1}$.

From this we can conjecture that e corresponding to the element $\hat{e}$ of an invariant subspace p can induce the $\sharp p$ and follows; the $\sharp p$ consists of all tensors of the form eF where F is an arbitrary tensor.

This definition is simpler than the former.

We now present the theorem.

Th. 2. Let p be an invariant subspace of c and β the corresponding $\sharp p$. If $p=p_1+p_2$, we have $\beta=\beta_1+\beta_2$ for the corresponding β's.

Pf) The generating unit $\hat{e}=\hat{e}_1+\hat{e}_2$ gives $eF=e_1F+e_2F$, the complete reduction of β.

Th. 3. Let p be an invariant subspace of c and $\beta=\sharp p$, then $p=\natural\beta$.

Pf) β is composed of all elements of the form

$$eF=\sum_\alpha{}' ec_\alpha E_\alpha=\sum_\alpha{}' c_\alpha(eE_\alpha)\equiv\sum_\alpha c_\alpha G_\alpha\equiv G$$

with E_α a basis for R^r.

The $\natural\beta$ contains all elements of the form

$$c_{\alpha,i_1\cdots i_r}g_\alpha[i_1\cdots i_r]\equiv b.$$

An arbitrary element of the group space a can be expressed as

$$a=\sum_{\alpha,(i)}c_{\alpha,i_1\cdots i_r}e_\alpha[i_1\cdots i_r]$$

then $a\hat{e}=b$ from the corresponding equation $eF=G$.

For the converse theorems, let β be an invariant subspace of R^r and $p=\natural\beta$.

Th. 4. $\beta=\sharp p$

The p contains all elements of the form

$$\sum_{\alpha,(i)}c_{\alpha,i_1\cdots i_r}e_\alpha[i_1\cdots i_r]$$

where (E_α) is a basis for β.

The generating unit $\hat{e}$ of p can be expressed as

$$\hat{e}=\sum_{\alpha,(i)}{}'\hat{e}_{\alpha,i_1\cdots i_r}e_\alpha[i_1\cdots i_r]$$

An element of $\sharp p$ is of the form $G=eF$ where

$$e_s=\hat{e}_{S^{-1}}=\sum{}_{\alpha,(k)}\hat{e}_{\alpha,k_1\cdots k_r}(S^{-1}E_\alpha)^{k_1\cdots k_r}$$
$$=\sum_{\alpha,(k)}{}'S(\hat{e}_{\alpha,k_1\cdots k_r})E_\alpha^{k_1\cdots k_r}$$

Then

$$G^{i_1\cdots i_r}=\sum_S{}'e_s(SF)^{i_1\cdots i_r}$$
$$=\sum_S\sum_{\alpha,(k)}S(\hat{e}_{\alpha,k_1\cdots k_r})E_\alpha^{k_1\cdots k_r}(SF)^{i_1\cdots i_r}$$
$$=\sum_{\alpha,(k)}{}'[\sum_S S(\hat{e}_{\alpha,k_1\cdots k_r})(SF)^{i_1\cdots i_r}]E_\alpha^{k_1\cdots k_r}$$
$$=\sum_{\alpha,(k)}{}'c_\alpha(i_1\cdots i_r\,;\,k_1\cdots k_r)E_\alpha^{k_1\cdots k_r}$$

where $c_\alpha(i_1\cdots i_r\,;\,k_1\cdots k_r)$ is a symmetric transformation coefficient.

Since β is invariant any element of $\sharp p$ is contained in β. Therefore $\sharp p\subset\beta$, and the converse is true. Therefore $\beta=\sharp p$.

Th. 5. If $\beta=\beta_1+\beta_2$, then $p=p_1+p_2$

Pf) The only part which remains to be proved is that p_1, p_2 are independent.

Let p^* be the intersection of p_1 and p_2.

By Th. 4., $\sharp p^*\subset\sharp p_1\subset\beta_1$ and $\sharp p^*\subset\beta_2$.

So $\sharp p^*$ is empty, and p^* is empty.

By now we show the complete analogy between the tensor space R^r and the group space c.

The irreducibility of one subspace in tensor space implies the irreducibility of the corresponding group subspace.

Therefore the manifold of totally symmetric tensors of R^r is irreducible since the corresponding group space is irreducible.

And the totally antisymmetric tensors also form an invariant irreducible subspace under the symmetric transformations.

V. Concluding Remarks

we have obtained the close analogy between the tensor space R^r induced on a vector space R and the group space of the symmetric permutation.

So if we obtain the reduction of group space π_r by some method[8] we naturally obtain the reduction of the corresponding tensor space.

The reduction of the group space can be accomplished by the use of the Young diagrams.

Correspondingly we naturally obtain the reduction of the tensor space.

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