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On nearly-countable compact space and locally nearly-compact

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〈Abstract〉

In this paper we define nearly-countable compact space, its properties and locally nearly-compact space. In section 2, we consider the relation of nearly-compact space, almostcountable compact space and countable compact space. In section 3 we study some properties of nearly-compact space and locally nearly-compact space.

Nearly-countable compact space와 locally nearly compact space에 관한 연구

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〈요 약〉

본 논문에서는 nearly-countable compact space를 정의하고 그들의 몇가지 성질과 locally nearly-compact space에 대하여 알아 보았다. 제 2 실에서는 nearly-countable compact space, almost-countable compact space, countable compact space를 비교하고 게 3 절에서는 nearly compact space를 이용한 locally nearly-compact space에 대하여 조사하였다.

I. Introduction

A topological space $(X, \mathcal{T})$ is said to be nearly-countable compact if every countable open cover of X has a finite subcollection, the interiors of the closures of which cover X; a topological space $(X, \mathcal{T})$ is said to be locally nearly-compact (2) if each point has an open neighborhood whose closure is nearly-compact subset of X. The purpose of this note is some notation of nearly-countable compact space and locally nearly-compact space and mapping. A set A is called regularly open (1) if it is the interior of its own closure or equivalently, if it is the interior of some closed set. A is called regularly-closed (1), if it is the closure of its own interior or equivalently, if it is closure of some open set. Throughout, $\bar{A}$ will denote the closure of a set A and $(A)^\circ$ will denote the interior of a set A.

II. Nearly-countable compact space

Definition 2.1. A space is said to be semi-regular if for each point x of the space and each open set U containg x, there is an open set V such that $x \in V \subset (\bar{V})^\circ \subset U$.

Theorem 2.1. A semi-regular and Lindelöf space is a nearly-countable compact if and only if it is countable compact.

Proof. Let$(X, \mathcal{T})$ be a semi-regular nearly-

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countable compact space, and $\mathscr{U}=\{U_\alpha: \alpha\in Z^+\}$ be any countable open cover of X. For each $x\in X$, there exists an $d_\alpha\in Z^+$ such that $x\in U_{dx}$. Since X is semi-regular, therefore there existsan open set G_x such that $x\in G_x\subset(\overline{G}_x)^\circ\subset U_{dx}$ X is nearly-countable compact there exists $\{C_x: x\in X\}$ such that $\cup\{(\overline{G_{xi}})^\circ: i=1, 2, \cdots, n\}=X$. Thus $X=\cup\{(\overline{G_{xi}})^\circ: i=1, 2, \cdots, n\}\subset\cup\{U_{xi}: i=1, 2, \cdots, n\}$. Hence X is countable compact. The converse is obviously, the class of nearly-countable compact space contains the class of countable compact space.

Definition 2.2. A topological space $(X, \mathscr{T})$ is said to be almost-countable compact if every countable open cover of X has a finite subcellection, whose closure cover of X.

Definition 2.3. A space $(X, \mathscr{T})$ is said to be almost-regular (1), if for every point $x\subset X$ and each neighborhood U of $x\in X$, there exists a neighborhood V of X such that $V\subset\overline{V}\subset(\overline{U})^\circ$.

Theorem 2.2. An almost-regular and Lindelöf space is an almost-countable compact space if and only if it is nearly-countable compact.

Proof. Let $\mathscr{U}=\{U_\alpha: \alpha\in Z^+\}$ be any regularly countable open cover of almos-tregular almost-countable compact.

For each $x\in X$, there exists an $\alpha_x\in Z^+$ such that $x\in \cup dx$. By almost-regularity there exists an open set V_x such that $x\in V_x\subset V_x\subset(U_{\alpha x})^\circ=U_{\alpha x}$. Let $V=\{V_x: x\subset X\}$ is an open cover of X which is an almost-countable compact space.

Therefore there exists a finite sub-family $\{V_{xi}: i=1, 2, \cdots, n\}$ of V such that $\cup\{V_{\alpha x}: i=1, 2, \cdots\cdots, n\}=X$. Thus X $\cup\{\overline{V_{\alpha x}}: i=1, 2, \cdots\cdots, n\}\subset\cup\{U_{\alpha xi}: i=1, 2, \cdots\cdots, n\}$.

Hence X is nearly-countable compact. The converse is obvious.

Definition 2.4 A mapping is said to be almost-continuous (1) if the inverse image of every regularly open set is open.

Theorem 2.3. The image of a countable compact space under an almost-continuous mapping is nearly-countable compact.

Proof. Let $f: X\to Y$ be an almost-continuous mapping of a countable compact space X onto a space Y. Let $V=\{V_\alpha: \alpha\in Z^+\}$ be a countable regularly open cover of Y.

Thus $\mathscr{V}=\{f^{-1}(V_\alpha): \alpha\in Z^+\}$ is a countable open cover of X. Since X is countable compact, therefore $\mathscr{U}$ has a finite sub-cover of $\{f^{-1}(V_{\alpha i}): i=1, 2, \cdots, n\}$, $\{V_{\alpha i}: i=1, 2, \cdots\cdots, n\}$ is a finite sub-cover of V. Hence Y is nearly countable compact.

Theorem 2.4 An almost-continuous image of an almost-countable compact space is almost-countable compact.

Proof. Let $f: X\longrightarrow Y$ bo an almost-continuous mapping of an almost-countable compact space X onto Y and let $\{U_\alpha: \alpha\in Z^+\}$ be any regularly countable open cover of Y. Then $\{f^{-1}(U_\alpha): \alpha\in Z^+\}$ is a countable open cover of X. Since X is almost-countable compact, therefore it has a finite sub-family $\{f^{-1}(U_{\alpha i}): i=1, 2, \cdots, n\}$ such that $\cup\{f^{-1}(U_{\alpha i}): i=1, 2, \cdots, n\}=X$. f being almost continuous, $\overline{f^{-1}(U_\alpha)}\subset f^{-1}(\overline{U}_\alpha)$. Hence $Y=\cup\{\overline{U_{\alpha i}}: i=1, 2, \cdots\cdots, n\}$. Thus Y is almost-countable compact.

Definition 2.5. A mapping $f: X\longrightarrow Y$ is said to be strongly continuous (4) if $f(\overline{A})\subset f(A)$ for every subset of A of X.

Theorem 2.5. The image of an almost-countable compact space under a strongly continuous mapping is countable compact.

Proof. Let $f: X\longrightarrow Y$ be a strongly continuous mapping of an almost-countable compact space X onto Y. Let $V=\{V_\alpha: \alpha\subset Z^+\}$ be any countable open cover of Y. Then $\mathscr{U}=\{f^{-1}(V_\alpha): \alpha\subset Z^+\}$ is a countable open cover of X. Since X is almost-countable compact space, therefore there exists a finite sub-family $\{f^{-1}(V_{\alpha i}): i=1, 2, \cdots, n\}$ of $\mathscr{U}$ such that $\cup\{f^{-1}(\overline{V}_{\alpha i}): i=1, 2, \cdots, n\}=X$, i.e. $\cup\{f^{-1}(V_{\alpha i}): i=1, 2, \cdots\cdots, n\}=X$ as f is strongly continuous. Hence $\{V_{\alpha i}: i=1, 2, \cdots\cdots, n\}$ is a finite sub-cover of V. Y is consequently countable compact.

Ⅲ. Locally nearly-compact space

Definition 3.1. Let $(X, \mathscr{T})$ be a topological space. A subset A of X is nearly-compact space if and only if for any cover of A by regularly open sets, there exists a finite subcover.

Theorem 3.1. A regularly closed subset of nearly-compact space is itself nearly-compact space.

Proof. Let C be nearly-compact space, B regularly closed and $B\subset C$. Let Q be a regularly open cover of B. Then $\mathscr{O}\cup\{X-B\}$ is a regularly open cover of C so there exists a finite subcollection of $\mathscr{O}$, say $\{O_i: i=1, 2, \cdots, n\}$ such that $C\subset(X-B)\cup\{O_i: i=1, 2, \cdots\cdots, n\}$. It follows that $B\subset\cup\{O_i: i=1, 2, \cdots\cdots, n\}$ so by Lemma 3.1, B is nearly-compact space.

Theorem 3.2. Let $(X, \mathscr{T})$ be a topological space. A subset B of X is nearly-compact space and O a regularly open set contained in B. Then $B-O$ is nearly-compact space.

Proof. Let $\mathscr{O}$ be a regularly open cover of $B-O$, $\mathscr{O}\cup\{O\}$ is a regularly open cover of B. Therefore, there exists a finite subcollection $\{O_i: i=1, 2, \cdots\cdots, n\}$ such that $B\subset O\ U\{O_i: i=1, 2, \cdots\cdots, n\}$ which implies $B-O$ is nearly-compact space.

Theorem 3.3. In a Hausdorff space $(X, \mathscr{T})$, let B be nearly-compact space. For any x in B and any regularly open set A such that $x\in A\subset B$, there is an open set V such that $x\in V\subset \bar{V}\subset A$.

Proof. Let $x\subset B$ and A any regularly open set such that $x\in A\subset B$. For each $y\in B-A$, there exists neighborhood $G_x{}^y$ and H_y such that $G_x{}^y\cap H_y=\phi$. Furthermore, we can assume each $G_x{}^y\subset A$.

The collection $\{H_y: y\in B-A\}$ is an open cover of the set $B-A$ which by Theorem 3.2, is nearly-compact space. Therefore, there exists a finite subcollection $\{H_{y_i}: i=1, 2, \cdots\cdots, n\}$ such that $B-A\subset\cup\{(\overline{H_{y_i}})^\circ: i=1, 2, \cdots, n\}=H$. Let $G=\cap\{G_x{}^{y_i}: i=1, 2, \cdots, n\}$. It then follows that $G\cap H=\phi$ so, since G and H are open, $\bar{G}\cap H=\phi$. Also, since B is closed and $G\subset A\subset B$, $\bar{G}\subset B$. Therefore, $B-A\subset B\cap H\subset B-\bar{G}$ which implies $\bar{G}\subset A$. Clearly then $x\in G\subset \bar{G}\subset A$ as specified.

Lemma 3.2. A finite union of sets nearly-compact space is nearly-compact space.

Proof. Let $B=\cup\{B_i: i=1, 2, \cdots\cdots, n\}$ where each B_i is nearly-compact. Let $\mathscr{O}$ be any open cover of B. Then $\mathscr{O}$ covers B_i for each i. Therefore, there exists a finite subcollection $\{O_j{}^i: j=1, 2, \cdots\cdots, m_i\}$ such that $B_i\subset U\{(\overline{O}_j{}^i)^\circ: j=1, 2, \cdots, m_i\}$. It follows that $B\subset U\{(\overline{O}_j{}^i)^\circ: j=1, 2, \cdots, m_i;\ i=1, 2, \cdots\cdots, n\}$.

Theorem 3.4. The following conditions are equivalent in Hausdorff space.

(a) X is locally nearly-compact.

(b) For each x in X and each neighborhood U of x, there is an open set V such that $\bar{V}$ is nearly-compact space and $x\in V\subset\bar{V}\subset(\bar{U})^\circ$.

(c) For each x in X and each neighborhood U of x, there is a regularly open set V is nearly-compact space such that $x\in V\subset\bar{V}\subset U$.

(d) For each set C nearly-compact space and regularly open $U\supset C$, there is an open V which is nearly-compact space and $C\subset V\subset\bar{V}\subset U$.

Proof. (a) implies (b). There is an open set W with $x\subset W\subset\bar{W}$ and $\bar{W}$ is nearly-compact space. The set $(U\cap\bar{W})^\circ$ is regularly open and is contained in $\bar{W}$. By Theorem 3.3, there exists an open set V such that $x\in V\subset(\overline{U\cap W})^\circ\subset(\bar{U})^\circ$. The set $\bar{V}$ is regularly closed and contained in $\bar{W}$ so by Theorem 3.1, $\bar{V}$ is nearly-compact space.

(b)$\Rightarrow$(c) is clearly.

(c)$\Rightarrow$(d). For each $c\in C$, find an open V_c such that $\bar{V}_c$ is nearly-compact space and $\bar{V}_c\subset U$. Since C is nearly-compact space, there exists a finite subcollection $\{V_{ci}: i=1, 2, \cdots\cdots, n\}$ such that $C\subset\cup\{(\bar{V}_{ci})^\circ: i=1, 2, \cdots, n\}=V$. Since $\bar{V}=\cup\{\bar{V}_{ci}: i=1, 2, \cdots, n\}$. $\bar{V}$ is the finite union of sets nearly-compact space so by Lemma 3.2, $\bar{V}$ is nearly-compact space. Clearly $C\subset V\subset \bar{V}\subset U$.

(d)$\Rightarrow$(a). A point is certainly nearly-compact space. Let X be U in (d).

Then for any x in X, the corresponding V in (d) is the neighborhood which is nearly-compact space.

Theorem 3.5. If B has an open neighborhood U whose closure is nearly-compact Then B also has a regularly open neighborhood V which closure is nearly-compact with $U \subset V \subset \bar{U}$.

Proof. If $B \subset U$ and $\bar{U}$ is nearly-compact the $B \subset (U)^{\circ} \subset (\overline{U})^{\circ-} = \bar{U}$. Therefore $(\bar{U})^{\circ}$ is the desired neighborhood ddesired.

Theorem 3.6. A semi-regular space is a locally nearly-compact space iff it is locally compact space.

Proof. Let $(X, \mathscr{T})$ be a semi-regular locally nearly-compact space, $x \in X$ and V is a semi-regular basic open set about x.

By Theorem 3.3 and 3.4, there is a regularly open neighborhood U of x with $x \in U \subset \bar{U} \subset V$ and $\bar{U}$ is nearly-compact space.

Since X is semi-regular space. Hence $\bar{U}$ is compact. It follows that $(x, \mathscr{T})$ is locally compact.

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