공학 연구논문집 제23권 제2호 pp. 49~59. 울산대학교
Journal of Engineering Research Vol. 23. No. 1. pp. 49~59. 1992.
University of Ulsan.

이산 vortex에 의하여 simulate된 vortex shedding과 유동 가속으로 인하여 원주에 작용되는 양력과 항력 계산식

이 동 기
조선 및 해양공학과

〈요 약〉

원주에서의 vortex 유출에 대한 이산화 vortex 해석을 할 때 사용할 수 있도록 원주에 작용되는 양력과 항력 계산식을 유도하였다. 이 식들은 유출된 vortex에 관련된 인수들로 나타내졌으며 확장된 Blasius 정리를 조작하여 구하여졌다. 원주 중심에 vortex가 있다고 보는가 없다고 보는가의 두 가지 가능한 Image system을 동시에 다루었다. 어느 경우건 현재까지 주목되지 않은 초생 vortex성장율을 포함하는 새로운 항이 이 계산식에 나타남을 보였다.

Formulas for the Calculation of Lift and Drag Exerted to a Circular Cylinder due to the Flow Acceleration and the Vortex Shedding Simulated by Discrete Vortices

Lee, Dong-Kee
Dept. of Naval Architecture and Ocean Engineering

〈Abstract〉

Formulas to calculate drag and lift exerted to a circular cylinder are derived for use in connection with the discrete vortex analysis of vortex shedding from the cylinder. The formulas are expressed in terms of the parameters concerned with the vortices and are obtained by manipulating the extended Blasius theorem. Two possibilities of the image system for a vortex outside the cylinder are covered simultaneously, the difference between these being whether the image system possesses a vortex at the center of the cylinder or not. In either case, the

formulas are shown to have a new term containing the growth rates of the so-called nascent vortices which has not been noticed so far.

Notations

U	:	time dependent uniform stream
a	:	radius of the circular cylinder
x, y	:	rectangular coordinates
z	:	complex coordinate, $z=x+iy$
t	:	time
i	:	the imaginary unit
	:	to imply image when used as a subscript
C	:	a closed contour coinciding with the cylinder circumference
φ	:	the complex velocity potential
Γ	:	vortex strength (positive counterclockwise)
M	:	the number of the shed discrete vortices
m	:	the number of the nascent vortices
u, v	:	x- and y-component of velocity respectively
ρ	:	density of the fluid
D, L	:	the drag and the lift, respectively, exerted to the cylinder

1. Introduction

The two-dimensional vortex shedding from a cylindrical body has been in recent years quite successfully simulated by the discrete vortex method. The vorticity in the boundary layer and the wake of the cylinder is represented by a system of concentrated point vortices. These vortices all created at the neighbourhood of the cylinder surface are convected as required by the flow field and in that way reveal the evolution of the typical vortex array in a similar form to the reality. The rotational flow of viscous fluid is thus modelled by the flow of ideal fluid with embedded vortices.

The staggered arrangement of the clusters of vortices exerts a drag of fluctuating magnitude and a nearly harmonically oscillating lift to the cylinder. These forces may be evaluated by the use of the extended Blasius theorem[1] which is in turn an application of the Bernoulli's theorem. Sarpkaya[2][3] applied this theorem to his discrete vortex shedding from a circular cylinder and obtained the formulas elegant and convenient for use. Lee[4] also considered derivation of these formulas and showed presence of the extra terms which do not exist in Sarpkaya's ones.

The formulas have different appearances depending on the form of the associated velocity potential. The velocity potential to describe the hypothesized flow field has dual manifestations according to the authors and still seems to stay at that stage. Clements[5], Sarpkaya[2], Sarpkaya and Garrison[6] and Laird[7] used

$$\varphi(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i}{2\pi} \sum_k \Gamma_k \left[\log(z - z_k) - \log(z - z_{ik}) + \log z \right] \tag{1}$$

while many others, including Sarpkaya himself in his later works, employed the expression

$$\varphi(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i}{2\pi} \sum_k \Gamma_k \left[\log(z - z_k) - \log(z - z_{ik}) \right] \tag{2}$$

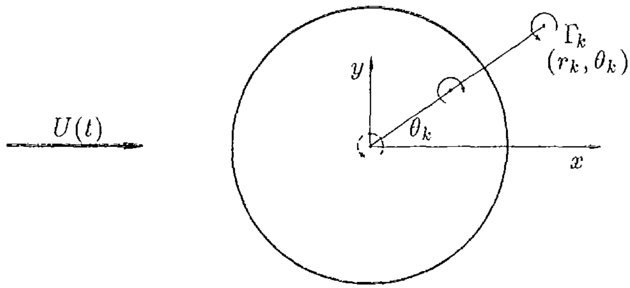
The point of dispute is whether the vortex should or should not be put at the center of the cylinder to complete the image system. Sarpkaya and Schoaff[8] offer the reason for their choice of eq.(2) rather than eq.(1) that 'there are no images at the center of the cylinder because the vortices have been shed from the cylinder and leave circulation opposite to their own on the body'. The existence of the starting vortex and the stopping vortex associated with the sudden start and the subsequent sudden stop of an aerofoil may justify this view. On the other hand, the situation speculated by bringing the cylinder in the flow field created by a vortex would clearly support the validity of eq.(1), the vortex being supposed to represent one of those created at the boundary layer.

However, it is not the purpose of the present paper to clarify which expression is logically legitimate although it may be stated that the eq.(2) produced better results in connection with the generated flow fields in the author's experience. It is the object of this paper to show that whichever expression may be chosen the growth rate of the so-called nascent vortex plays a role in determining the magnitude of the flow-induced forces. This source of the force has never been noticed so far and, in the author's opinion, the calculations without inclusion of the effect of these terms should be necessarily modified.

2. Representation of the flow field

2.1 Statements of the flow situation

A circular cylinder of radius a is introduced in an otherwise undisturbed flow with its timedependent velocity $U(t)$ in the positive x-direction. The fluid is assumed to be inviscid and the strength of a vortex, once released into the flow field, does not change. A system of vortices are supposed to exist outside the cylinder, typical one of which is as shown in the Fig.1.



radius a

Fig. 1. vortices around the cylinder placed in a uniform time-dependent flow

The origin of the coordinate system is placed at the center of the cylinder. Both rectangular and polar coordinates are used as convenient.

2.2 The velocity

The complex velocity potentials given by eq.(1) and eq.(2) may be combined as

$$\varphi(z) = U(t)\left(z + \frac{a^2}{z}\right) - \frac{i}{2\pi} \sum_{k=1}^{M(t)} \Gamma_k \left[\log(z - z_k) - \log(z - z_{ik}) + \lambda \log z \right] \quad (3)$$

$$\text{where } \lambda = \begin{cases} 1, & \text{for eq.(1)} \\ 0, & \text{for eq.(2).} \end{cases}$$

The velocity at an arbitrary point not coinciding with the position of a vortex is then given by

$$\begin{aligned} u - iv &= \frac{d\varphi}{dz} \\ &= U\left(1 - \frac{a^2}{z^2}\right) - \frac{i}{2\pi} \sum_{k=1}^{M(t)} \Gamma_k \left(\frac{1}{z - z_k} - \frac{1}{z - z_{ik}} + \frac{\lambda}{z} \right) \end{aligned} \quad (4)$$

If the velocity at the position of a vortex, say the m-th, is in question, it is to be calculated from

$$\begin{aligned} u_m - iv_m &= u(z_m) - iv(z_m) \\ &= U\left(1 - \frac{a^2}{z_m^2}\right) - \frac{i}{2\pi} \sum_{\substack{k=1 \\ k \neq m}}^{M(t)} \Gamma_k \left(\frac{1}{z_m - z_k} - \frac{1}{z_m - z_{ik}} + \frac{\lambda}{z_m} \right) \\ &\quad - \frac{i}{2\pi} \Gamma_m \left(- \frac{1}{z_m - z_{im}} + \frac{\lambda}{z_m} \right) \end{aligned} \quad (5)$$

2.3 The image vortices

The image vortices to satisfy the boundary condition of zero normal velocity are, as is well known, specified by Milne-thomson's circle theorem. z_{ik} in the eq.(3), (4) and (5) represents the position of the image of the k-th vortex and is related to the position of the k-th vortex z_k by

$$z_{ik} = a^2/\bar{z}_k \tag{6}$$

Its strength is, according to the theorem, given by

$$\Gamma_{ik} = -\Gamma_k$$

3. Derivation of the force formulas

3.1. The extended Blasius theorem

The extended Blasius theorem[1] takes the following form in the present case

$$D - iL = \frac{1}{2} i\rho \oint_C \left(\frac{d\varphi}{dz}\right)^2 dz + i\rho\frac{\partial}{\partial t}\oint_C \bar{\varphi}\, d\bar{z} \tag{7}$$

Denote the first and the second term as follows to consider them separately

$$D_1 - iL_1 = \frac{1}{2} i\rho \oint_C \left(\frac{d\varphi}{dz}\right)^2 dz \tag{8}$$

$$D_2 + iL_2 = -i\rho\frac{\partial}{\partial t}\oint_C \varphi\, dz \tag{9}$$

$$\text{with} \quad D = D_1 + D_2, \quad L = L_1 + L_2 \tag{10}$$

3.1.1. The first integral($D_1 - iL_1$)

Noticing that there are a finite number of singularities outside the cylinder, the first integral may, from the Cauchy-Goursat theorem, by evaluated by

$$D_1 - iL_1 = \frac{1}{2} i\rho \left[\oint_{C_\infty} \left(\frac{d\varphi}{dz}\right)^2 dz - \sum_{k=1}^{M}\oint_{C_k}\left(\frac{d\varphi}{dz}\right)^2 dz\right] \tag{11}$$

where C_∞ : a closed contour of large radius enclosing the cylinder and all the shed vortices

C_k : a closed contour of small radius enclosing the k-th vortex only.

On the circle of large radius C_∞, taking the following relation

$$\frac{1}{z - z_k} - \frac{1}{z - z_{ik}} = \frac{z_k - z_{ik}}{(z - z_k)(z - z_{ik})} \tag{12}$$

in eq.(4) into consideration, the integrand has the property

$$(\frac{d\varphi}{dz})^2 = U^2 - \lambda\frac{iU\Gamma_\Sigma}{\pi}\frac{1}{z} + O(|z|^{-2}) \qquad \text{as } |z| \rightarrow \infty$$
$$\text{where} \qquad \Gamma_\Sigma = \sum_k \Gamma_k \tag{13}$$

and therefore application of the Cauchy integral formula yields

$$\oint_{C_\infty} (\frac{d\varphi}{dz})^2 dz = 2\pi i\,(-\lambda\frac{iU\Gamma_\Sigma}{\pi}) = 2\lambda U\Gamma_\Sigma \tag{14}$$

On the other hand, at any point on the small circle enclosing the l-th vortex, the integrand can be written as follows

$$(\frac{d\varphi}{dz})^2 = -\frac{\Gamma_l^2}{4\pi^2(z-z_l)^2} - \frac{i}{\pi}\frac{\Gamma_l}{z-z_l}[U(1-\frac{a^2}{z^2}) - \frac{i}{2\pi}\sum_{\substack{k=1\\k\neq l}}^{M}\Gamma_k(\frac{1}{z-z_k}-\frac{1}{z-z_{ik}}+\frac{\lambda}{z}) - \frac{i}{2\pi}\Gamma_l(-\frac{1}{z-z_{il}}+\frac{\lambda}{z})] + H(z) \tag{15}$$

where $H(z)$:an analytic function on and within the circle c_l enclosing the l-th vortex.

Note that the expression within the brackets coincides with that for the velocity of the l-th vortex given by eq.(5), as z approaches z_l. Thus we obtain

$$\oint_{C_k} (\frac{d\varphi}{dz})^2 dz = 2\Gamma_k(u_k - iv_k) \tag{16}$$

Then, collecting the results in eq.(14) and in eq.(16), we have

$$D_1 - iL_1 = i\rho[\lambda U\Gamma_\Sigma - \sum_{k=l}^{M}\Gamma_k(u_k - iv_k)] \tag{17}$$

3.1.2. The second integral(D_2+iL_2)

Inserting eq.(3) into eq.(9), we have the following expression

$$D_2 + iL_2 = -i\rho \frac{\partial}{\partial t} \oint_C \{ U(t) \left(z + \frac{a^2}{z}\right) - \frac{i}{2\pi} \sum_{k=1}^{M(t)} \Gamma_k \left[\log(z - z_k) - \log(z - z_{ik}) + \lambda \log z \right] \} dz \tag{18}$$

In the process of derivation with respect to time, it is to be noted that the number of vortices increases by as many as the number of the nascent vortices, that is

$$M(t + \Delta t) = M(t) + m \tag{19}$$

This assertion comes from the fact that the interval of the time step of introducing vortices is an arbitrary parameter. Then, when manipulated from the definition of a derivative, the following expression is finally obtained

$$D_2 + iL_2 = 2\pi a^2 \rho \frac{dU}{dt} - i\rho \sum_{k=1}^{M} \Gamma_k (u_{ik} + iv_{ik}) + i\rho \sum_{k=1}^{m} \frac{d\Gamma_{nk}}{dt} \left(1 - \frac{a}{r_{nk}} - \lambda\right) a e^{i\theta_{nk}} \tag{20}$$

where $r_{nk}e^{i\theta_{nk}}$ denotes the position of the k-th nascent vortex and $(u_{ik}+iv_{ik})$ the velocity of the image of the k-th vortex. The details of the derivation are shown in the Appendix.

3.2. The force formulas

Collecting eq.(17) and eq.(20) together, we obtain the formulas for the drag and the lift as follows

$$D = 2\pi a^2 \rho \frac{dU}{dt} - \rho \sum_{k=1}^{M} \Gamma_k (v_k - v_{ik}) - \rho a \sum_{k=1}^{m} \frac{d\Gamma_{nk}}{dt} \left(1 - \frac{a}{r_{nk}} - \lambda\right) \sin\theta_{nk} , \tag{21}$$

$$L = \rho \sum_{k=1}^{M} \Gamma_k (u_k - u_{ik} - \lambda U) + \rho a \sum_{k=1}^{m} \frac{d\Gamma_{nk}}{dt} \left(1 - \frac{a}{r_{nk}} - \lambda\right) \cos\theta_{nk} . \tag{22}$$

The term $d\Gamma_{nk}/dt$ represents the growth rate of the nascent vortices and should depend on the principle of determining the strength of the nascent vortices.

Frequently, the strength of the nascent vortices is determined so that the tangential component of velocity vanishes at the point on the cylinder surface and directly below the nascent vortex. If this principle is employed, since the normal component of velocity is everywhere zero on the cylinder surface on account of the use of the image vortices, the following system of m simultaneous equations can be set up for m unknown Γ's.

$$U(1-e^{2i\theta_{nl}}) - \frac{i}{2\pi}\sum_{k=1}^{M}\Gamma_k\left[\frac{1}{a e^{i\theta_{nl}} - z_k} - \frac{1}{a e^{i\theta_{nl}} - z_{ik}} + \frac{\lambda}{a e^{i\theta_{nl}}}\right]$$

$$-\frac{i}{2\pi}\sum_{k=1}^{M}\Gamma_{nk}\left[\frac{1}{a e^{i\theta_{nl}} - r_{nk} e^{i\theta_{nk}}} - \frac{1}{a e^{i\theta_{nl}} - \frac{a^2}{r_{nk}} e^{i\theta_{nk}}} + \frac{\lambda}{a e^{i\theta_{nl}}}\right] = 0, \qquad (23)$$

for $l = 1, 2, \ldots, m$.

In the matrix notation, this system of simultaneous equations becomes

$$\mathrm{A}\,\Gamma_n = \mathrm{B}$$
$$\text{or} \qquad \Gamma_n = \mathrm{C}\,\mathrm{B}, \qquad \text{with } \mathrm{C} = \mathrm{A}^{-1} \qquad (24)$$

$$\text{where} \quad A_{lk} = \frac{i}{2\pi}\left(\frac{1}{a e^{i\theta_{nl}} - r_{nk} e^{i\theta_{nk}}} - \frac{1}{a e^{i\theta_{nl}} - \frac{a^2}{r_{nk}} e^{i\theta_{nk}}} + \frac{\lambda}{a e^{i\theta_{nl}}}\right), \qquad (25)$$

$$B_l(t) = U(t)(1-e^{2i\theta_{nl}}) - \frac{i}{2\pi}\sum_{k=1}^{M}\left[\frac{1}{a e^{i\theta_{nl}} - z_k(t)} - \frac{1}{a e^{i\theta_{nl}} - z_{ik}(t)} + \frac{\lambda}{a e^{i\theta_{nl}}}\right],$$

for $k, l = 1, 2, \ldots, m$. (26)

Then the strength of the l-th nascent vortex is

$$\Gamma_{nl} = \sum_{k=1}^{m} C_{lk} B_k \qquad (27)$$

And the growth rate of the nascent vortex is given by

$$\frac{d\Gamma_{nl}}{dt} = \sum_{k=1}^{M} C_{lk}\frac{dB_k}{dt} \qquad (28)$$

It is to be noted that the matrix A is indepent of time t and so is the matrix C.

4. Conclusion

The terms containing the growth rate of the nascent vortices in the present force formulas are what have so far been neglected. Obviously, inclusion of the effect of these terms would bring about not negligible consequences on the values of lift and drag estimated without it.

Sarpkaya's force formulas[2] do have terms containing the rate of change of vortex strengths. But this should be interpreted as the rate of change of vortex strengths of those vortices already released into the flow field. In other words, it is concerned with the decay of vortex strengths. The vorticity shed into a viscous fluid decays with time and hence it may well be justified to incorporate a decaying mechanism into the present way of modelling the real vortex shedding process by the discrete vortices embedded in an ideal fluid. However, this means introduction of another arbitrary parameter for the decaying rate into the method, which makes the present author to state that it is preferrable to manage the method to work without such parameter.

The formulas naturally depend on which of eq.(1) or eq.(2) is used for the velocity potential. The difference is wrapped up in the parameter λ. The decision of which expression is more appropriate to describe the flow field is not the topic of the present investigation but, rather, it may be suggested that the present force formulas contain some implicative ingredients to deserve consultation in the process of making such decision.

References

1. Milne-Thomson, L.M., Theoretical Hydrodynamics, The Macmillan Company, New York, N.Y., 1960.
2. Sarpkaya, T., Lift, Drag, and Added-Mass Coefficients for a Circular Cylinder Immersed in a Time-Dependent Flow, J. of Appl. Mech., Trans. ASME, pp.13-15, March 1963.
3. Sarpkaya, T., An Analytical Study of Separated Flow About Circular Cylinders, J. of Basic Engineering, Trans. ASME, pp.511-520, Dec. 1968.
4. Lee, D. K., Lift and Drag of a Circular Cylinder by the Discrete Vortex Method, J. of the Soc. Nav. Arch., Korea, vol 27, no. 2, pp.40-46, Jun. 1990.
5. Clements, R. R., Flow representation, including separated regions, using discrete vortices, AGARD LS no.86 on Computational Fluid Dynamics, Paper no. 5, 1977.
6. Sarpkaya, T. and Garrison, G. J., Vortex Formtion and Resistance in Unsteady Flow, J. Appl. Mech., Trans. ASME, pp. 16-24, Mar. 1963.
7. Laird, A. D. K., Eddy formation behind circular cylinders, J. of hydraulic div., Proc. ASCE, vol. HY6, pp.763-775, Jun. 1971.
8. Sarpkaya, T. and Schoaff, R.L., Inviscid Model of Two-Dimensional Vortex Shedding by a Circular Cylinder, AIAA Journal, vol. 17, no. 11, pp.1193-1200, Nov. 1979.

Appendix

A.1. The time derivative of eq.(18)

Keeping in mind the fact that the positions of vortices are changing and that the number of vortices should increase by the number of nascent vortices m, the derivation may proceed as follows

$$D_2 + iL_2 = -i\rho\frac{\partial}{\partial t}\oint_C [U(t)(z + \frac{a^2}{z}) - \frac{i}{2\pi}\sum_{k=1}^{M(t)} \Gamma_k (\log[z - z_k(t)] - \log[z - z_{ik}(t)] + \lambda\log z)]\, dz$$

$$= -i\rho\frac{dU}{dt}\oint_C (z + \frac{a^2}{z})\, dz + \frac{\rho}{2\pi}\lim_{\Delta t \to 0}\frac{1}{\Delta t}\oint_C \{\sum_{k=1}^{M+m}\Gamma_k(\log[z - z_{ik}(t+\Delta t)] - \lambda\log z) - \sum_{k=1}^{M}(\log[z - z_{ik}(t)] - \lambda\log z)\}\, dz$$

$$= 2\pi\rho a^2\frac{dU}{dt} - \frac{\rho}{2\pi}\sum_{k=1}^{M}\Gamma_k\oint_C \frac{1}{z - z_{ik}}\frac{dz_{ik}}{dt}\, dz + \frac{\rho}{2\pi}\sum_{k=1}^{m}\frac{d\Gamma_{nk}}{dt}\oint_C [\log(z - z_{ink}) - \lambda\log z]\, dz$$

$$= 2\pi\rho a^2\frac{dU}{dt} - i\rho\sum_{k=1}^{M}\Gamma_k(u_{ik} + iv_{ik}) + i\rho\sum_{k=1}^{m}\frac{d\Gamma_{nk}}{dt}(1 - \lambda - \frac{a}{r_{nk}})\, a\, e^{i\theta_{nk}}$$

where $u_{ik} + iv_{ik} = \dfrac{dz_{ik}}{dt}$: complex velocity of the image of the k-th vortex

$z_{ink} = \dfrac{a^2}{r_{nk}} e^{i\theta_{nk}}$: position of the image of the k-th nascent vortex

The final expression is what is denoted as eq.(20). The details of the integration of the logarithmic function in the above deribvation follows.

A.2. Integration of the logarithmic function

Since a complex logarithmic function is multi-valued, it is necessary to consider a particular branch taking the branch cut at the radial line emanating through the position of the relevant vortex. Then

$$\oint_C [\log(z - z_{ink}) - \lambda \log z]\, dz$$

$$= [(z - z_{ink}) \log(z - z_{ink}) - (z - z_{ink}) - \lambda(z \log z - z)]_{a e^{i\theta_{nk}}}^{a e^{i(\theta_{nk} + 2\pi)}}$$

$$= [(a e^{i\theta} - z_{ink})(\log r(\theta) + i\alpha(\theta) + 2k_1\pi i) - (a e^{i\theta} - z_i)$$

$$- \lambda(a e^{i\theta}(\log a + i\theta + 2k_2\pi i) - a e^{i\theta})]_{\theta=\theta_{nk}}^{\theta=\theta_{nk}+2\pi}$$

$$= 2\pi i(a e^{i\theta_{nk}} - z_{ink}) - \lambda 2\pi i\, a e^{i\theta_{nk}}$$

$$= 2\pi i(1 - \lambda - \frac{a}{r_{nk}}) a e^{i\theta_{nk}}$$

where k_1, k_2 ; arbitrary integer constants

in which the polar representation $r(\theta)\, e^{i\alpha(\theta)} = z - z_{ink}$ has been introduced at the intermediate step and $z_{ink} = (a^2 / r_{nk})\, e^{i\theta_{nk}}$ is substituted at the final stage.