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Soft set theory 에서의 Heyting algebras 의 filters

(Filters of Heyting algebras on soft set theory)

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Filters of Heyting algebras on soft set theory

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먼저 학부과정 때부터 석사과정을 거쳐 박사과정까지 긴 시간동안 항상 부족한 저에게 수학이라는 학문의 가치와 깊이를 일깨워주시고 항상 기다려 주시며 큰 가르침을 주신 이동수 교수님께 진심어린 감사를 드립니다. 아울러 항상 진심어린 조언과 격려로 힘을 주시고 많은 학문적 가르침과 충고를 아끼지 않으신 박철환 교수님께도 깊은 감사를 드립니다. 교수님께서 베풀어 주신 넓은 아량과 배려 덕분으로 여기까지 올 수 있었다고 생각합니다. 교수님께서 보여주신 학자로서의 모습과 후진을 양성하는 모습 하나하나 가슴에 새겨 더욱 정진하는데 주춧돌로 삼겠습니다.

아울러 우리 대수학 팀 김종헌 선생님, 최용섭 선생님, 박갑진 선생님께도 진정어린 감사를 드립니다. 이후에도 항상 좋은 관계가 유지되기를 기원하며 노력하겠습니다.

또한 논문심사과정에서 진심어린 충고와 격려 및 수고를 아끼지 않으셨던 울산대학교 박건 교수님, 강태호 교수님, 장창림 교수님, 이양 교수님께 감사의 말씀을 드리며, 알찬 강의로 학문적인 소양을 기르는데 도움을 주신 장선영 교수님, 심인보 교수님, 추상목 교수님, 장준명 교수님, 이현호 교수님께도 감사의 말씀을 드립니다.

현재까지 학원운영을 하면서 강의와 여러 학원업무를 병행하며 공부를 해야 했던 현실에서 많은 배려와 격려를 아끼지 않으셨던 정치송 과학원장님 이하 여러 도움을 주신 선생님들께 고마움의 말씀을 전합니다.

살아온 날 보다 같이 살아갈 날이 더 많이 남은, 자신을 희생하면서 가정과 내조에 혼신의 힘을 다해 임해준 나의 아내 유승현에게 평생의 동반자로서 감사의 뜻을 표합니다. 아울러 늘 바쁘다는 핑계로 좋은 아빠가 되어 주지 못했는데도 불구하고 항상 웃으며 건강하게 자라나는 우리 건후와 가빈이 그리고 아직은 뱃속에 있는 릴리(태명)와 기쁨을 같이 나누고 싶습니다. 항상 부족한 사위를 자랑스러워하시며 격려해 주시는 장인, 장모님께 감사의 말씀을 드립니다.

끝으로 오늘이 있기까지 기다려주시고, 항상 걱정해주시며, 언제나 믿어주신 어머님께 이 박사학위 논문을 바칩니다.

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박 동 민

< CONTENTS >

Filters of Heyting algebras on soft set theory

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1 Introduction

In mathematics, Heyting algebras is special bounded lattices that constitutes a generalization of Boolean algebras. In the 19th century, Luitzen Brouwer founded the mathematical philosophy of intuitionism. Intuitionism is based on the idea that mathematics is a creation of the mind and believed that a statement could only be demonstrated by a direct proof. Arend Heyting, a student of Brouwer's, formalized this thinking into his namesake algebras (Heyting algebra). Heyting algebras have played an important role and have its comprehensive applications in many aspects including genetic code of biology, dynamical systems and algebraic theory [2, 3, 4, 5, 6, 7, 8, 14].

The complexities of modeling uncertian data in economics, engineering, environment and many other fields cannot successfully use classical methods because of various uncertainties typical for those problems.

To overcome these difficulties, Molodtsov [17] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Maji et al. [15] also studied several operations on the theory of soft sets. Since then, soft set theory has wide range of application in economics, engineering, environment, information science, inteligence system and algebraic structure [10, 11, 16].

In this paper, we define the intersection soft filter (IS-filter), Boolean intersectional soft filter (Boolean IS-filter), and ultra inetersectional soft filter(Ultra IS-filter) and investigates related properties. We discuss characterizations of IS-filter and Boolean IS-filter and consider relations between IS-filters and Boolean IS–filters.

In section 2, we recall the definition of heyting algebra and investigate several properties of Heyting algebras. Also we introduce filter and soft set.

In section 3, we introduce the definition of IS-filter and investigate several properties.

In section 4, we introduce the concept of Boolean IS-filter and investigate some of the properties. Also we investigate the relation between IS-filter and Boolean IS-filter.

In section 5, we introduce the concept of ultra IS-filter and investigate some of the properties. Also we introduce the concept of prime IS-filter and investigate the relation between ultra IS-filter and prime Boolean IS-filter.

2 Preliminaries

In this section, we review some definitions and properties that will be useful in our results. At first we introduce the definition of a Heyting algebra.

Definition 2.1. [1] Heyting algebra is defined to be a bounded lattice $\mathcal{H}$ such that for any pair of elements $x, y \in \mathcal{H}$, there is the largest element $z \in \mathcal{H}$ such that $z \wedge x \leq y$. This element is denoted by $x \to y$ and is called an implication. The operation which sends each element x the element $x' = x \to 0$ is called a negation.

The definition of implication is equivalent to the existence of an element $x \to y$ such that

$$z \wedge x \leq y \iff z \leq x \to y$$

Some elementary properties of Heyting algebras are summarized by the following.

Proposition 2.2. [1] *For elements x, y, z in a Heyting algebra:*

(hp1) $x \wedge (x \to y) \leq y$,

(hp2) $x \wedge y \leq z \iff y \leq x \to z$,

(hp3) $x \leq y \iff x \to y = 1$,

(hp4) $y \leq x \to y$,

(hp5) $x \leq y \Longrightarrow z \to x \leq z \to y$ and $y \to z \leq x \to z$, (hp6) $x \to (y \to z) = (x \wedge y) \to z$,

(hp7) $x \wedge (y \to z) = x \wedge \{(x \wedge y) \to (x \wedge z)\}$,

(hp8) $x \wedge (x \to y) = x \wedge y$,

(hp9) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(hp10) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Proof. (hp1) $(x \to y) \leq (x \to y) \Leftrightarrow (x \to y) \wedge x \leq y \Leftrightarrow x \wedge (x \to y) \leq y$.

(hp2) ($\Rightarrow$) $x \wedge y = y \wedge x \leq z \Leftrightarrow y \leq x \to z$.

($\Leftarrow$) $y \leq x \to z \Leftrightarrow y \wedge x \leq z \Leftrightarrow x \wedge y \leq z$.

(hp3) $x \to y = 1 \Leftrightarrow 1 \leq x \to y \Leftrightarrow 1 \wedge x \leq y \Leftrightarrow x \leq y$.

(hp4) $x \wedge y \leq y \Leftrightarrow y \wedge x \leq y \Leftrightarrow y \leq x \to y$.

(hp5) $z \wedge (z \to x) \leq x \leq y$

$\Rightarrow (z \to x) \wedge z \leq y$

$\Rightarrow z \to x \leq z \to y$,

and $x \leq y \Rightarrow x \wedge (y \to z) \leq y \wedge (y \to z) = y \wedge z \leq z$

$\Rightarrow (y \to z) \wedge x \leq z$

$\Rightarrow (y \to z) \leq (x \to z)$.

(hp6) $(x \wedge y) \wedge (x \to (y \to z)) = y \wedge (x \wedge (x \to (y \to z))) \leq y \wedge (y \to z) \leq z$

$\Leftrightarrow (x \to (y \to z)) \wedge (x \wedge y) \leq z$, so $x \to (y \to z) \leq (x \wedge y) \to z$.

Conversely, $(x \wedge y) \wedge ((x \wedge y) \to z) \leq z \Leftrightarrow (x \wedge ((x \wedge y) \to z)) \wedge y \leq z$,

so $x \wedge ((x \wedge y) \to z) \leq y \to z$,

and hence $(x \wedge y) \to z \leq x \to (y \to z)$.

(hp7) $x \wedge (x \to y) \leq x$, and $(x \wedge y) \wedge x \wedge (y \to z) \leq x \wedge z$,

so $x \wedge (y \to z) \leq (x \wedge y) \to (x \wedge z)$.

Hence $x \wedge (y \to z) \leq x \wedge \{(x \wedge y) \to (x \wedge z)\}$.

Conversely, $x \wedge ((x \wedge y) \to (x \wedge z)) \leq x$ and $(y \wedge x) \wedge ((x \wedge y) \to (x \wedge z)) \leq x \wedge z \leq z$,

so $x \wedge ((x \wedge y) \to (x \wedge z)) \leq y \to z$.

Hence $x \wedge \{(x \wedge y) \to (x \wedge z)\} \leq x \wedge (y \to z)$.

(hp8) $x \wedge (x \to y) \leq x$ and $x \wedge (x \to y) \leq y$,

so $x \wedge (x \to y) \leq x \wedge y$.

Conversely, $x \wedge y \leq x$ and $x \wedge y \leq x \to y$ so $x \wedge y \leq x \wedge (x \to y)$.

(hp9) $x \leq x \vee y$ and $y \leq x \vee y$ implies

$(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$,

so $(x \vee y) \to z \leq (x \to z) \wedge (y \to z)$.

Conversely, $(x \vee y) \wedge (x \to z) \wedge (y \to z) \leq \{x \wedge (x \to z)\} \vee \{y \wedge (y \to z)\} \leq z \vee z = z$ so $(x \to z) \wedge (y \to z) \leq (x \vee y) \to z$.

(hp10) $y \wedge z \leq y$ and $y \wedge z \leq z$

implies $x \to (y \wedge z) \leq x \to z$ and $x \to (y \wedge z) \leq x \to z$, so $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$.

Conversely, $y \leq x \to y$ implies $x \wedge y \leq x \wedge (x \to y)$,

so $x \wedge (x \to y) \wedge (x \to z) \leq x \wedge y \wedge (x \to z) \leq y \wedge z$.

Hence $(x \to y) \wedge (x \to z) \leq x \to (y \wedge z)$.

□

The following corollary is an immediate consequence of Proposition 2.2.

Corollary 2.3. *For elements x, y, z in a Heyting algebra:*

(hp11) $x \to (y \to z) = y \to (x \to z)$,

(hp12) $x \to 1 = 1, 1 \to x = x, x \to x = 1$,

(hp13) $x \to (y \to x) = 1$,

(hp14) $(x \vee y) \leq (x \to y) \to y$.

Proof. (hp11) Using (hp6) we have $x \to (y \to z) = (x \wedge y) \to z = (y \wedge x) \to z = y \to (x \to z)$.

(hp12) $x \leq 1 \Rightarrow x \to 1 = 1$.

By (hp8),we have $1 \to x = 1 \wedge (1 \to x) = 1 \wedge x = x$ and

$x \leq x \Rightarrow x \to x = 1$.

(hp13) Using (hp11) and (hp12), we have

$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1$.

(hp14) Using (hp6) and (hp9), we get $(x \vee y) \to ((x \to y) \to y)$

$$=(x \to ((x \to y) \to y)) \wedge (y \to ((x \to y) \to y))$$
$$=((x \to y) \to (x \to y)) \wedge ((x \to y) \to (y \to y))$$
$$=1 \wedge ((x \to y) \to 1)$$
$$=1 \wedge 1$$
$$=1,$$

and so $(x \vee y) \leq (x \to y) \to y$ by (hp3). □

Here are some well known examples.

Example 2.4. [8] (1) Every Boolean algebra is a Heyting algebra and every Heyting algebra is a distributive lattice.

(2) Every bounded chain lattice $\mathcal{H}$ is a Heyting algebra. Indeed, for any $a, b \in \mathcal{H}$

$$a \to b := \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

In what follows let $\mathcal{H}$ denote an Heyting-algebra unless otherwise specified.

Some kinds of filters in a Heyting algebra is defined as follows.

Definition 2.5. [7] A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called a *filter* of $\mathcal{H}$ if it satisfies

(1) $(\forall x, y \in \mathcal{H}) \quad (x \in \mathcal{F}, x \leq y \ \Rightarrow \ y \in \mathcal{F})$,

(2) $(\forall x, y \in \mathcal{H}) \quad (x, y \in \mathcal{F}, x \wedge y \in \mathcal{F})$.

Proposition 2.6. [7] A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called a *filter* of $\mathcal{H}$ if it satisfies

(1) $1 \in \mathcal{F}$,

(2) $(\forall x, y \in \mathcal{H})\ \ (x \in \mathcal{F}, x \to y \in \mathcal{F} \ \Rightarrow\ y \in \mathcal{F})$.

Definition 2.7. [7] Let $\mathcal{F}$ be a filter of $\mathcal{H}$. $\mathcal{F}$ is called a Boolean filter of $\mathcal{H}$ if it satisfies $(x \wedge x') \in \mathcal{F}$ for all $x \in \mathcal{H}$.

Definition 2.8. [7] Let $\mathcal{F}$ be a filter of $\mathcal{H}$. $\mathcal{F}$ is called an ultra filter of $\mathcal{H}$ if it satisfies $x \in \mathcal{F}$ or $x' \in F$ for all $x \in \mathcal{H}$.

Molodtsov [17] introduced the concept of soft set as a new mathematical tool, and Çağman et al. [10] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$

Definition 2.9. [10, 17] A soft set f_A of E (over U) is defined to be the set of

$$f_A := \{f_A(x) \in \mathscr{P}(U) : x \in E, f_A(x) = \emptyset \text{ if } x \notin A.\},$$

where f_A is a mapping given by $f_A : E \to \mathscr{P}(U)$.

Example 2.10. Let $U = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ be a universal set consisting of a set of six cars under consideration and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ a set of parameters with respect to U, where each parameters $e_i, i = 1, 2, 3, ..., 7$ stands for expensive, cheap, sedan, wagon, sport utility vehicle, in good repair, in bad repair, respectively and $A = \{e_1, e_3, e_6\} \subseteq E$. A soft set f_A describes the attractiveness of the cars, such that $f_A(e_1) = \{c_1, c_3\}$, $f_A(e_3) = \{c_4, c_5, c_6\}$ and $f_A(e_6) = \{c_1, c_3, c_5\}$. Then the soft set f_A is a parameterized family $\{f_A(e_1), f_A(e_3), f_A(e_6)\}$ of subsets of U.

Remark 2.11. [17] Zadeh's fuzzy set may be considered as a special case of the soft set. Let A be a fuzzy set, and μ_A be the membership function of the fuzzy set A, that is μ_A is a mapping of U into [0,1]. Let us consider the family of α-level sets for function μ_A

$$f_{[0,1]}(\alpha) = \{x \in U \mid \mu_A(x) \geq \alpha\}, \alpha \in [0,1].$$

If we konw the family $f_{[0,1]}$, we can find the function $\mu_A(x)$ by mean of the following formulae:

$$\mu_A(x) = sup\{\alpha : \alpha \in [0,1], x \in f_{[0,1]}(\alpha)\}$$

Thus, every Zadeh's fuzzy set A may be considered as the soft set $f_{[0,1]}$.

Definition 2.12. [10, 17] For a soft set f_A of E over U and a subset τ of U, the set

$$i_A(f_A; \tau) = \{x \in A \mid f_A(x) \supseteq \tau\}$$

is called the τ-*inclusive set* of f_A.

3 Intersectional soft filter (IS-filter)

In this section, we introduce the concept of IS-filter in Heyting algebras, and investigate their properties.

Definition 3.1. A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is called an *IS-filter* of $\mathcal{H}$ if it satisfies:

(f1) $(\forall x, y \in \mathcal{H}) \quad (x \leq y \;\Rightarrow\; f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}(y))\,,$

(f2) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \wedge y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(y))\,.$

We provide characterizations of an IS-filter.

Proposition 3.2. *A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies:*

(f3) $(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x))\,,$

(f4) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \rightarrow y))\,.$

Proof. Suppose that $f_{\mathcal{H}}$ of $\mathcal{H}$ is an IS-filter of $\mathcal{H}$. Since $x \leq 1$ for all $x \in \mathcal{H}$, it follows from Definition3.1(f1) that

$$f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x)$$

for all $x \in \mathcal{H}$. This proves (f3) hold. By (hp1), we have $x \wedge (x \rightarrow y) \leq y$. Hence

$$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \wedge (x \rightarrow y)).$$

By Definition3.1(f2),

$$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \wedge (x \rightarrow y)) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \rightarrow y).$$

This proves (f4).

Conversely, assume that $f_{\mathcal{H}}$ satisfies conditions (f3) and (f4). Let $x, y \in \mathcal{H}$ such that $x \leq y$ then $x \to y = 1$ by (h3). By condition (f4) and (f3), we have

$$\begin{aligned} f_{\mathcal{H}}(y) &\supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y) \\ &= f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(1) \\ &= f_{\mathcal{H}}(x), \end{aligned}$$

which implies, $f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}(y)$. This prove (f1).

By (hp6) and (hp12), we have $x \to (y \to (x \wedge y)) = (x \wedge y) \to (x \wedge y) = 1$. By Definition 3.1 (f2), we have

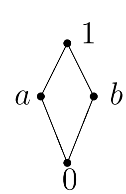
$$\begin{aligned} f_{\mathcal{H}}(x \wedge y) &\supseteq f_{\mathcal{H}}(y) \cap f_{\mathcal{H}}(y \to (x \wedge y)) \\ &\supseteq f_{\mathcal{H}}(y) \cap (f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to (y \to (x \wedge y)))) \\ &= f_{\mathcal{H}}(y) \cap (f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(1)) \\ &= f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(y), \end{aligned}$$

for all $x, y \in \mathcal{H}$. This proves (f2), and so $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

The following example shows that an IS-filter exists.

Example 3.3. Let $\mathcal{H} = \{0, a, b, 1\}$ be a set with the following Cayley table and Hasse diagram.

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1



Then $\mathcal{H}$ is a Heyting algebra. Let $f_{\mathcal{H}}$ be a soft set over $U = \mathbf{Z}$ in $\mathcal{H}$ given as follows:

$$f_{\mathcal{H}}(x) = \begin{cases} 2\mathbf{Z} & \text{if} \quad x \in \{a, 1\} \\ 2\mathbf{N} & \text{if} \quad \text{otherwise} \end{cases}$$

1. (f1) is clear.

2. We will show that $(f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y))$.

   1) $y = 0$

$$2N = f_{\mathcal{H}}(0) \supseteq \begin{cases} f_{\mathcal{H}}(0) \cap f_{\mathcal{H}}(0 \to 0) = 2N \cap f_{\mathcal{H}}(1) = 2N \cap 2Z \\ f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(a \to 0) = 2Z \cap f_{\mathcal{H}}(b) = 2Z \cap 2N \\ f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(b \to 0) = 2N \cap f_{\mathcal{H}}(a) = 2N \cap 2Z \\ f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(1 \to 0) = 2Z \cap f_{\mathcal{H}}(0) = 2Z \cap 2N \end{cases}$$

   2) $y = a$

$$2Z = f_{\mathcal{H}}(a) \supseteq \begin{cases} f_{\mathcal{H}}(0) \cap f_{\mathcal{H}}(0 \to a) = 2N \cap f_{\mathcal{H}}(1) = 2N \cap 2Z \\ f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(a \to a) = 2Z \cap f_{\mathcal{H}}(1) = 2Z \cap 2Z \\ f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(b \to a) = 2N \cap f_{\mathcal{H}}(a) = 2N \cap 2Z \\ f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(1 \to a) = 2Z \cap f_{\mathcal{H}}(a) = 2Z \cap 2Z \end{cases}$$

   3) $y = b$

$$2N = f_{\mathcal{H}}(b) \supseteq \begin{cases} f_{\mathcal{H}}(0) \cap f_{\mathcal{H}}(0 \to b) = 2N \cap f_{\mathcal{H}}(1) = 2N \cap 2Z \\ f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(a \to b) = 2Z \cap f_{\mathcal{H}}(b) = 2Z \cap 2N \\ f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(b \to b) = 2N \cap f_{\mathcal{H}}(1) = 2N \cap 2Z \\ f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(1 \to b) = 2Z \cap f_{\mathcal{H}}(b) = 2Z \cap 2N \end{cases}$$

4) $y = 1$

$$2Z = f_{\mathcal{H}}(1) \supseteq \begin{cases} f_{\mathcal{H}}(0) \cap f_{\mathcal{H}}(0 \to 1) = 2N \cap f_{\mathcal{H}}(1) = 2N \cap 2Z \\ f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(a \to 1) = 2Z \cap f_{\mathcal{H}}(1) = 2Z \cap 2Z \\ f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(b \to 1) = 2N \cap f_{\mathcal{H}}(1) = 2N \cap 2Z \\ f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(1 \to 1) = 2Z \cap f_{\mathcal{H}}(1) = 2Z \cap 2Z \end{cases}$$

Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$.

Theorem 3.4. *A soft set $f_{\mathcal{H}}$ in $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if*

(f5) $(\forall a, b, c \in \mathcal{H}) \quad (a \to (b \to c) = 1 \Longrightarrow f_{\mathcal{H}}(c) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Let $a, b, c \in \mathcal{H}$ be such that $a \to (b \to c) = 1$. By (hp3), we have $a \leq b \to c$. Then $f_{\mathcal{H}}(b \to c) \supseteq f_{\mathcal{H}}(a)$ by (f1), and so

$$f_{\mathcal{H}}(c) \supseteq f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(b \to c) \supseteq f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(a).$$

Conversely, let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ satisfying (f5). By $x \leq 1$ and (hp12) we have $x \to (x \to 1) = 1$ it follows from (f5) that

$$f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x)$$

for all $x \in \mathcal{H}$. Using (hp12), we know that $(x \to y) \to (x \to y) = 1$ for all $x, y \in \mathcal{H} = 1$. It follows from (f5) that

$$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y)$$

for all $x, y \in \mathcal{H}$. Therefore $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Corollary 3.5. *A soft set $f_{\mathcal{H}}$ in $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if* (f6) $(\forall a, b, c \in \mathcal{H}) \quad ((a \wedge b) \leq c \Longrightarrow f_{\mathcal{H}}(c) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b))$.

Proof. Using (hp2) and (hp3), we have $(a \wedge b) \to c = (a \to (b \to c)) = 1$. Therefore Corollary is valid by Theorem 3.4. □

Theorem 3.6. *Let $f_{\mathcal{H}}$ be a soft set in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies conditions* (f3) *and*

(f7) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((x \to (y \to z)) \cap f_{\mathcal{H}}(y))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Since $x \to (y \to z) = y \to (x \to z)$, we have

$$(x \to (y \to z)) \to (y \to (x \to z)) = 1$$

by (hp3). From Theorem 3.4, we have

$$f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (y \to z)) \cap f_{\mathcal{H}}(y)$$

for all $x, y, z \in \mathcal{H}$. Convesely, suppose that $f_{\mathcal{H}}$ satisfies condition (f3) and (f7). Putting $x = 1$ in (f7) and using (hp12), we have

$$f_{\mathcal{H}}(z) = f_{\mathcal{H}}(1 \to z) \supseteq f_{\mathcal{H}}(1 \to (y \to z)) \cap f_{\mathcal{H}}(y) = f_{\mathcal{H}}(y \to z) \cap f_{\mathcal{H}}(y)$$

for all $x, y \in \mathcal{H}$. Therefore $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Theorem 3.7. *Let $f_{\mathcal{H}}$ be a soft set in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of H if and only if it satisfies conditions* (f3) *and*

(f8) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (y \to z))) \cap f_{\mathcal{H}}(x \to y))$.

Proof. Assume $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Since $y \wedge (y \to z) \leq z$, we have

$$x \to z \geq x \to ((y \to z) \wedge y) = (x \to (y \to z)) \wedge (x \to y).$$

by (hp5) and (hp10). From Corollary 3.5, we have

$$f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((x \to (y \to z)) \cap f_{\mathcal{H}}(x \to y).$$

Conversely, suppose that $f_{\mathcal{H}}$ satisfies conditions (f1) and (f8). Taking $x = 1$ in (f8) and using (f2), we have

$$f_{\mathcal{H}}(z) \supseteq f_{\mathcal{H}}(y \to z) \cap f_{\mathcal{H}}(y)$$

for all $x, y \in \mathcal{H}$. Hence $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Theorem 3.8. *Let $f_{\mathcal{H}}$ be an IS-filter in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies conditions* (f3) *and*

(f9) $(\forall x, y, z \in \mathcal{H})$ $(f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(y \to z))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. By (hp6), (hp8), and (hp11), we have

$$\begin{aligned}
(x \to y) \to ((y \to z) \to (x \to z)) &= (y \to z) \to ((x \to y) \to (x \to z)) \\
&= (y \to z) \to (((x \to y) \wedge x) \to z) \\
&= (y \to z) \to ((x \wedge y) \to z) \\
&= (y \to z)(x \to (y \to z)) \\
&= x \to ((y \to z) \to (y \to z)) \\
&= x \to 1 \\
&= 1.
\end{aligned}$$

It follows from Theorem 3.4, we have we have

$$f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(y \to z)\}.$$

This proves (f9) hold. Suppose that $f_{\mathcal{H}}$ satisfies conditions (f3) and (f9). Obviously $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x)$. Taking $x = 1$ in (f9) and using (hp12), we have $f_{\mathcal{H}}(z) \supseteq f_{\mathcal{H}}(y \to z) \cap f_{\mathcal{H}}(y)$ for all $x, y \in f_{\mathcal{H}}$. This proves (f4), and so $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ by Proposition 3.2

□

Theorem 3.9. *Let $f_{\mathcal{H}}$ be a soft set in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies the following conditions:*

(f10) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(x))$,

(f11) $(\forall x, a, b \in \mathcal{H}) \quad (f_{\mathcal{H}}((a \to (b \to x)) \to x) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Using (hp13), we get

$$f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(x \to (y \to x)) \cap f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x)$$

for all $x, y \in \mathcal{H}$. By (hp11) and $a \to ((a \to (b \to x)) \to (b \to x)) = (a \to (b \to x)) \to (a \to (b \to x)) = 1$, we get $a \subseteq ((a \to (b \to x)) \to (b \to x))$. It follows from (f3) that

$$f_{\mathcal{H}}((a \to (b \to x)) \to (b \to x)) \supseteq f_{\mathcal{H}}(a).$$

By Theorem 3.6 we have

$$f_{\mathcal{H}}((a \to (b \to x)) \to x) \supseteq f_{\mathcal{H}}((a \to (b \to x)) \to (b \to x)) \cap f_{\mathcal{H}}(b) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b).$$

Conversely, let $f_{\mathcal{H}}$ be an IS-filter in $\mathcal{H}$ satisfying conditions (f10) and (f11). If we take $y = x$ in (f11), then

$$f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \to x) \supseteq f_{\mathcal{H}}(x)$$

for all $x \in \mathcal{H}$. Using (f11), we obtain

$$f_{\mathcal{H}}(y) = f_{\mathcal{H}}(1 \to y) = f_{\mathcal{H}}(((x \to y) \to (x \to y)) \to y) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x)$$

for all $x, y \in \mathcal{H}$. Therefore $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Theorem 3.10. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$. Then the following are equivalent:*

(f12) $(\forall x, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (z' \to z))$,

(f13) $(\forall x, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$,

(f14) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(y \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y)$,

(f15) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq \{f_{\mathcal{H}}(x \to (z' \to y)) \cap f_{\mathcal{H}}(y \to z)\})$.

Proof. (f12) ⇒ (f13) Assume that $f_{\mathcal{H}}$ satisfies the condition (f12) and let $x, y, z \in \mathcal{H}$. Using (hp5) and (hp11), we know that

$$x \to z \leq z' \to (x \to z) = x \to (z' \to z).$$

Using (f1), we have

$$f_{\mathcal{H}}(x \to z) \subseteq f_{\mathcal{H}}(x \to (z' \to z)).$$

Therefore $f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$.

(f13) ⇒ (f14) Assume that $f_{\mathcal{H}}$ satisfies the condition (f13) and let $x, y, z \in \mathcal{H}$. Since $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$, we have

$$f_{\mathcal{H}}(x \to (z' \to z)) \supseteq f_{\mathcal{H}}(y \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y).$$

Using (f13), then we have

$$f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z)) \supseteq f_{\mathcal{H}}(y \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y).$$

(f14) $\Rightarrow$ (f15) Assume that $f_{\mathcal{H}}$ satisfies the condition (f14) and let $x, y, z \in \mathcal{H}$. By (hp5) and $(z' \to y) \leq ((y \to z) \to (z' \to z))$ then we have

$$x \to (z' \to y) \leq x \to ((y \to z) \to (z' \to z)).$$

It follows from (f1) that

$$f_{\mathcal{H}}(x \to ((y \to z) \to (z' \to z))) \supseteq f_{\mathcal{H}}(x \to (z' \to y)).$$

Using (f14), (hp11), and (f3), we have

$$\begin{aligned} f_{\mathcal{H}}(x \to z) &\supseteq f_{\mathcal{H}}((y \to z) \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y \to z) \\ &= f_{\mathcal{H}}(x \to ((y \to z) \to (z' \to z))) \cap f_{\mathcal{H}}(y \to z) \\ &\supseteq f_{\mathcal{H}}(x \to (z' \to y)) \cap f_{\mathcal{H}}(y \to z) \end{aligned}$$

for all $x, y \in \mathcal{H}$.

(f15) $\Rightarrow$ (f12) Assume that $f_{\mathcal{H}}$ satisfies the condition (f12) and let $x, y, z \in \mathcal{H}$. Taking $y = z$ in condition (f15) and using (f3), we obtain

$$\begin{aligned} f_{\mathcal{H}}(x \to z) &\supseteq f_{\mathcal{H}}(x \to (z' \to z)) \cap f_{\mathcal{H}}(z \to z) \\ &= f_{\mathcal{H}}(x \to (z' \to z)) \cap f_{\mathcal{H}}(1) \\ &= f_{\mathcal{H}}(x \to (z' \to z). \end{aligned}$$

Therefore $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (z' \to y))$.

□

The relation between IS-filter and its inclusive set is as follows:

Theorem 3.11. *A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if the nonempty τ-inclusive set $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$.*

Proof. Suppose that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ and for each $\tau \in \mathscr{P}(U)$ be such that $i_{\mathcal{H}}(f_{\mathcal{H}};\tau) \neq \emptyset$, then there exists $a \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ such that $f_{\mathcal{H}}(a) \supseteq \tau$.

By (f3) we have $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(a) \supseteq \tau$ and $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Let $x, y \in \mathcal{H}$ be such that $x \to y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ and $x \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Then $f_{\mathcal{H}}(x \to y) \supseteq \tau$ and $f_{\mathcal{H}}(x) \supseteq \tau$. It follows from (f4) that

$$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) \supseteq \tau,$$

that is, $y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Thus $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)\,(\neq \emptyset)$ is a filter of $\mathcal{H}$ by Proposition 2.6.

Conversely, suppose that τ-inclusive set $i_{H}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$ with $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)\,(\neq \emptyset)$. For any $x \in \mathcal{H}$, let $f_{\mathcal{H}}(x) = \tau$. Then $x \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Since $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$, hence $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. It follows that

$$f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x) = \tau.$$

Let $x, y \in \mathcal{H}$ such that $f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) = \tau$. Then

$$x, x \to y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau).$$

Since τ-inclusive set $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$, then we have $y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. It follows that

$$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) = \tau.$$

Therefore $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ by Proposition 3.2.

□

Theorem 3.12. *If $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$, then the set*

$$\Gamma_a := \{x \in \mathcal{H} \mid f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a)\}$$

is a filter of $\mathcal{H}$ for every $a \in \mathcal{H}$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter. For any $x \in \mathcal{H}$, since $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x)$, then $1 \in \Gamma_a$. Let $x, y \in \mathcal{H}$ be such that $x \in \Gamma_a$ and $x \to y \in \Gamma_a$. Then

$$f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a) \;\; \textit{and} \;\; f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a).$$

It folllow from (f1) that

$$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a).$$

Hence $y \in \Gamma_a$, and so Γ_a is a filter of $\mathcal{H}$. □

Theorem 3.13. *Let $a \in \mathcal{H}$ and let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$. Then*

(1) *If Γ_a is a filter of $\mathcal{H}$, then $f_{\mathcal{H}}$ satisfies the following condition:*

(f12) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a) \;\Rightarrow\; y \in \Gamma_a)$.

(2) *If $f_{\mathcal{H}}$ satisfies* (f1) *and* (f12)*, then Γ_a is a filter of $\mathcal{H}$.*

Proof. (1) Assume that Γ_a is a filter of $\mathcal{H}$. Let $x, y \in \mathcal{H}$ be such that $f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a)$. Then we have the following consequence

$$x \to y \in \Gamma_a \;\; \text{and} \;\; x \in \Gamma_a.$$

Since Γ_a is a filter, we have $y \in \Gamma_a$.

(2) Suppose that $f_{\mathcal{H}}$ satisfies (f3) and (f12). From (f1) it follows that $1 \in \Gamma_a$. Let $x, y \in \mathcal{H}$ be such that $x \in \Gamma_a$ and $x \to y \in \Gamma_a$. We have

$$f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a) \text{ and } f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a).$$

This implies that $f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a)$. By the assumed condition (f12), we get $y \in \Gamma_a$. Therefore Γ_a is a filter of $\mathcal{H}$ by Proposition 2.6. □

4 Boolean intersectional soft filter (Boolean IS-filter)

In this section, we introduce the concept of Boolean IS-filter and investigate some of the properties.

Definition 4.1. An IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$ is said to be Boolean IS-filter if the following assertion is valid.

$$(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)) \, .$$

Remark 4.2. Every Boolean IS-filter is IS-filter of $\mathcal{H}$, but the converse may not be true a shown in the following example.

Example 4.3. Let $\mathcal{H} = [0, 1]$ and define $\wedge, \vee$ and implication $\rightarrow$ on $\mathcal{H}$ as follows:

$$\begin{cases} x \wedge y & = \min\{x, y\}, \\ x \vee y & = \max\{x, y\} \end{cases} \quad x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y \end{cases}$$

for all $x, y \in \mathcal{H}$. Then $\mathcal{H}$ is a Heyting-algebra. Let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ in which

$$f_{\mathcal{H}}(x) := \begin{cases} \tau & \text{if } x \in [0.5, 1], \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\tau (\neq \emptyset) \in \mathscr{P}(U)$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$.

But it is not a Boolean IS-filter of $\mathcal{H}$ over U since $f_{\mathcal{H}}(1) = \tau$ and

$$f_{\mathcal{H}}(\frac{1}{3} \vee \frac{1}{3}') = f_{\mathcal{H}}(\frac{1}{3} \vee (\frac{1}{3} \rightarrow 0)) = f_{\mathcal{H}}(\frac{1}{3} \vee 0) = f_{\mathcal{H}}(\frac{1}{3}) = \emptyset$$

The following proposition serve as an useful satrting point in this chapter.

Proposition 4.4. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$, then the following are equivalent:*

(1) $(\forall x, z \in \mathcal{H})\ \ f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$,

(2) $(\forall x \in \mathcal{H})\ \ f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x' \to x)$,

(3) $(\forall x, y \in \mathcal{H})\ \ f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}((x \to y) \to x)$,

(4) $(\forall x, y \in \mathcal{H})\ \ f_{\mathcal{H}}(x) = f_{\mathcal{H}}((x \to y) \to x)$,

(5) $(\forall x, y, z \in \mathcal{H})\ \ f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(z \to ((x \to y) \to x)) \cap f_{\mathcal{H}}(z)$.

Proof. (1) $\Rightarrow$ (2) Assume that $f_{\mathcal{H}}$ satisfies the condition (1) and let $x \in \mathcal{H}$. Using condition (1), we have

$$f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1 \to x) = f_{\mathcal{H}}(1 \to (x' \to x)) = f_{\mathcal{H}}(x' \to x).$$

(2) $\Rightarrow$ (3) Since $x' \leq x \to y$, $(x \to y) \to x \leq x' \to x$, and so

$$f_{\mathcal{H}}(x' \to x) \supseteq f_{\mathcal{H}}((x \to y) \to x).$$

Thus, from (2), we can deduce that $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x' \to x) \supseteq f_{\mathcal{H}}((x \to y) \to x)$.

(3) $\Rightarrow$ (4) On the other hand, since $x \leq (x \to y) \to x$, we have

$$f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}((x \to y) \to x).$$

Thus, we can get

$$f_{\mathcal{H}}(x) = f_{\mathcal{H}}((x \to y) \to x).$$

(4) $\Rightarrow$ (5) Since $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$, then

$$f_{\mathcal{H}}((x \to y) \to x) \supseteq f_{\mathcal{H}}(z \to ((x \to y) \to x)) \cap f_{\mathcal{H}}(z).$$

It follows from (4) that

$$f_{\mathcal{H}}(x) = f_{\mathcal{H}}((x \to y) \to x) \supseteq f_{\mathcal{H}}(z \to ((x \to y) \to x)) \cap f_{\mathcal{H}}(z).$$

(5) $\Rightarrow$ (1) Since $z \leq x \to z$, we have $(x \to z)' \leq z'$ and $z' \to (x \to z) \leq (x \to z)' \to (x \to z)$. Thus, we have

$$f_{\mathcal{H}}((x \to z)' \to (x \to z)) \supseteq f_{\mathcal{H}}(z' \to (x \to z)).$$

It follows from (5) that

$$\begin{aligned} f_{\mathcal{H}}(x \to z) &\supseteq f_{\mathcal{H}}(1 \to (((x \to z) \to 0) \to (x \to z))) \cap f_{\mathcal{H}}(1) \\ &= f_{\mathcal{H}}((x \to z)' \to (x \to z)) \cap f_{\mathcal{H}}(1) \\ &\supseteq f_{\mathcal{H}}((x \to z)' \to (x \to z)) \\ &\supseteq f_{\mathcal{H}}(z' \to (x \to z)), \end{aligned}$$

which implies $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(z' \to (x \to z))$. Therefore, it follows from Theorem 3.10 that $f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(z' \to (x \to z))$.

□

In the following theorem, we have a characterization of Boolean IS-filters.

Theorem 4.5. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$, then the following are equivalent:*

(1) *$f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$,*

(2) $(\forall x, z \in \mathcal{H}) \quad f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$.

Proof. (1) $\Rightarrow$ (2) Suppose that $f_{\mathcal{H}}$ is a Boolean IS-filter and let $x, y \in \mathcal{H}$.

Using (f2) we have

$$\begin{aligned} f_{\mathcal{H}}(x \to z) &\supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)) \cap f_{\mathcal{H}}(z \vee z') \\ &\supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)) \cap f_{\mathcal{H}}(1) \\ &\supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)). \end{aligned}$$

From (hp9), (hp11), (hp12), and Definition 3.1, we get

$$\begin{aligned} (z \vee z') \to (x \to z) &= (z \to (x \to z)) \wedge (z' \to (x \to z)) \\ &= (x \to (z \to z)) \wedge (z' \to (x \to z)) \\ &= (x \to 1) \wedge (z' \to (x \to z)) \\ &= 1 \wedge (z' \to (x \to z)) \\ &= z' \to (x \to z) = x \to (z' \to z). \end{aligned}$$

Thus

$$f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)) = f_{\mathcal{H}}(x \to (z' \to z)).$$

(2) $\Rightarrow$ (1) Assume that $f_{\mathcal{H}}$ satisfies (2). Using Theorem 3.10 (3) and (hp12), we have

$$f_{\mathcal{H}}((x' \to x) \to x) = f_{\mathcal{H}}((x' \to x) \to (x' \to x)) = f_{\mathcal{H}}(1).$$

Using (hp5), (hp9), (hp11), and (hp12), we have

$$\begin{aligned} (x' \to x) \to x &\le (x' \to x) \to (x \vee x') \\ &= (1 \wedge (x' \to x)) \to (x \vee x') \\ &= ((x \to x) \wedge (x' \to x)) \to (x \vee x') \\ &= ((x \vee x') \to x) \to (x \vee x'). \end{aligned}$$

It follow from Definition 3.1 and Proposition 4.4 that

$$\begin{aligned} f_{\mathcal{H}}(1) &= f_{\mathcal{H}}((x' \to x) \to x) \\ &\subseteq f_{\mathcal{H}}(((x \vee x') \to x) \to (x \vee x')) \\ &= f_{\mathcal{H}}(x \vee x'), \end{aligned}$$

and so $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is a Boolean IS-filter. □

Combining Theorem 3.10, Proposition 4.4, and Theorem 4.5, we have the following result.

Theorem 4.6. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$. Then the following are equivalent:*

(1) *$f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$,*

(2) $(\forall x, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$,

(3) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}((x \to y) \to x))$,

(4) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq \{f_{\mathcal{H}}(x \to (z' \to y)) \cap f_{\mathcal{H}}(y \to z)\})$.

We now give an equivalent condition for a Boolean IS-filter.

Some properties of Heyting algebras can be observed in the following.

Lemma 4.7. *In Heyting algebra $\mathcal{H}$, the following are hold:*

(1) $(\forall x, y, z \in \mathcal{H}) \quad (x \to y \leq (y \to z) \to (x \to z))$

(2) $(\forall x, y, z \in \mathcal{H}) \quad (x \to y \leq (z \to x) \to (z \to y))$

(3) $(\forall x, y \in \mathcal{H}) \quad ((x \to y) \to y \leq (x \to (x \to y)) \to (x \to y))$

Proof. (1) Since $x \wedge y \leq y$, we have $x \wedge (x \rightarrow y) \leq x \wedge y \leq y$ by (hp8). It follows from (hp8) that

$$(x \wedge (x \rightarrow y)) \wedge (y \rightarrow z) \leq y \wedge (y \rightarrow z) \leq (y \wedge z) \leq z,$$

and so from (hp2)

$$(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z.$$

Thus, we have

$$x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z).$$

(2) Since $z \wedge x \leq x$, we have $z \wedge (z \rightarrow x) \leq z \wedge x \leq x$ by (hp8). It follows from (hp5) and (hp6) that

$$\begin{aligned} x \rightarrow y &\leq (z \wedge (z \rightarrow x)) \rightarrow y \\ &= ((z \rightarrow x) \wedge z) \rightarrow y \\ &\leq (z \rightarrow x) \rightarrow (z \rightarrow y). \end{aligned}$$

(3) Using (hp12) and (hp6), we get

$$\begin{aligned} (x \rightarrow y) \rightarrow y \leq 1 &= (x \rightarrow y) \rightarrow (x \rightarrow y) \\ &= ((x \wedge x) \rightarrow y) \rightarrow (x \rightarrow y) \\ &= (x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y). \end{aligned}$$

□

Theorem 4.8. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$. Then the following are equivalent:*

(1) *$f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$,*

(2) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(((x \rightarrow y) \rightarrow y) \rightarrow x) \supseteq f_{\mathcal{H}}(y \rightarrow x))$,

(3) $(\forall x, y, z \in \mathcal{H})\ \ (f_{\mathcal{H}}(((x \to y) \to y) \to x) \supseteq f_{\mathcal{H}}(z) \cap f_{\mathcal{H}}(z \to (y \to x)))$,

(4) $(\forall x, y \in \mathcal{H})\ \ (f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}((x \to y) \to x))$.

Proof. (1) $\Rightarrow$ (2) Suppose that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. Since $x \leq ((x \to y) \to y) \to x$ we have

$$(((x \to y) \to y) \to x) \to y \leq x \to y$$

by Lemma 4.7. Using Lemma 4.7 and (hp 11), we get

$$\begin{aligned}
&((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \\
&\geq (x \to y) \to (((x \to y) \to y) \to x) \\
&= ((x \to y) \to y) \to ((x \to y) \to x) \\
&\geq y \to x,
\end{aligned}$$

and so

$$f_{\mathcal{H}}((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \supseteq f_{\mathcal{H}}(y \to x)$$

for all $x, y \in H$ by Definition 3.1 (f1). It follows from Proposition 4.4 that

$$\begin{aligned}
f_{\mathcal{H}}(((x \to y) \to y) \to y) &\supseteq f_{\mathcal{H}}(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \\
&\supseteq f_{\mathcal{H}}(y \to x).
\end{aligned}$$

(2) $\Rightarrow$ (3) Assume that the condition (2) holds in $\mathcal{H}$ and let $x, y \in \mathcal{H}$. Since $f_{\mathcal{H}}$ is an IS-filter, we have

$$f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(z) \cap (z \to (y \to x)).$$

By appling to (2), we get

$$\begin{aligned}
f_{\mathcal{H}}(((x \to y) \to y) \to x) &\supseteq f_{\mathcal{H}}(y \to x) \\
&\supseteq f_{\mathcal{H}}(z) \cap (z \to (y \to x)).
\end{aligned}$$

(3) $\Rightarrow$ (4) Assume that $f_{\mathcal{H}}$ satisfies the condition (3) and that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Since $x \to (((x \to y) \to y) = (x \to y) \to (x \to y) = 1$, we have

$$x \leq (x \to y) \to y.$$

Using (hp5) and (hp6), we get

$$\begin{aligned}((x \to y) \to x) &\leq (x \to y) \to ((x \to y) \to y)\\ &= ((x \to y) \wedge (x \to y)) \to y\\ &= (x \to y) \to y.\end{aligned}$$

By Definition 3.1 (f1), we have

$$f_{\mathcal{H}}(x \to (((x \to y) \to y)) \subseteq f_{\mathcal{H}}((x \to y) \to y).$$

By Lemma 4.7 (3), we have $((x \to y) \to y \leq (x \to (x \to y)) \to (x \to y))$.
By (hp5), $(x \to (x \to y)) \to (x \to y)) \to x \leq ((x \to y) \to y) \to x$.
By the condition (3),

$$\begin{aligned}f_{\mathcal{H}}((x \to y) \to x) &\subseteq f_{\mathcal{H}}(((x \to (x \to y)) \to (x \to y)) \to x)\\ &= f_{\mathcal{H}}(((x \to y) \to y) \to x).\end{aligned}$$

Hence

$$f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}((x \to y) \to y) \cap f_{\mathcal{H}}(((x \to y) \to y) \to x)$$

By Proposition 3.2 (f3),

$$f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}(x).$$

Since $x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = 1$, we have

$$x \leq (x \to y) \to y.$$

Using (hp5) and (hp6), we get

$$\begin{aligned}((x \to y) \to x) &\leq (x \to y) \to ((x \to y) \to y) \\ &= ((x \to y) \wedge (x \to y)) \to y \\ &= (x \to y) \to y.\end{aligned}$$

By Definiton 3.1 (f1), we have

$$f_{\mathcal{H}}(x \to (((x \to y) \to y)) \subseteq f_{\mathcal{H}}((x \to y) \to y).$$

By Lemma 4.7 (3), we have $((x \to y) \to y \leq (x \to (x \to y)) \to (x \to y))$.
By (hp5), $(x \to (x \to y)) \to (x \to y)) \to x \leq ((x \to y) \to y) \to x$.
By the condition (2),

$$\begin{aligned}f_{\mathcal{H}}((x \to y) \to x) &\subseteq f_{\mathcal{H}}(((x \to (x \to y)) \to (x \to y)) \to x) \\ &= f_{\mathcal{H}}(((x \to y) \to y) \to x).\end{aligned}$$

Hence

$$f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}((x \to y) \to y) \cap f_{\mathcal{H}}(((x \to y) \to y) \to x).$$

By Proposition 3.2 (f4),

$$f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}(x).$$

(4) $\Rightarrow$ (1) By Theorem 4.6.

□

In the following theorem, we give relationship between IS-filters and its inclusive set is shown as follows.

Theorem 4.9. *A soft set $f_{\mathcal{H}}$ on $\mathcal{H}$ is a Boolean IS-filter of $\mathcal{H}$ if and only if the nonempty τ-inclusive set $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ on $\mathcal{H}$ is a Boolean filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$.*

Proof. Suppose that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. Let $\tau \in \mathscr{P}(U)$ with $i_{\mathcal{H}}(f_{\mathcal{H}};\tau) \neq \emptyset$. Then $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$ by Theorem 3.11. Hence, $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$, and so $\tau \subseteq f_{\mathcal{H}}(1)$. For all $x \in \mathcal{H}$. It follows from Definition 4.1 that

$$\tau \subseteq f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \vee x')$$

and so that $x \vee x' \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Therefore $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a Boolean filter of $\mathcal{H}$

Conversely suppose that $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a Boolean filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$ with $i_{\mathcal{H}}(f_{\mathcal{H}};\tau) \neq \emptyset$. Then $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$, and so $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Note that $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Since $i_{\mathcal{H}}(f_{\mathcal{H}}; f_{\mathcal{H}}(1))$ is a Boolean filter of $\mathcal{H}$, we get

$$x \vee x' \in i_{\mathcal{H}}(f_{\mathcal{H}}; f_{\mathcal{H}}(1))$$

for all $x \in \mathcal{H}$. Hence $f_{\mathcal{H}}(x \vee x') \supseteq f_{\mathcal{H}}(1)$. This implies that $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. □

In the following theorem, we establish the extension property of a Boolean IS-filter.

Theorem 4.10. (Extension property) *Let $f_{\mathcal{H}}$ and $g_{\mathcal{H}}$ be IS-filters of $\mathcal{H}$ such that $f_{\mathcal{H}}(1) = g_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(x) \subseteq g_{\mathcal{H}}(x)$ for all $x \in \mathcal{H}$. If $g_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$, then so is $f_{\mathcal{H}}$.*

Proof. Assume that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. Then $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$ for all $x \in \mathcal{H}$. Hence

$$f_{\mathcal{H}}(x \vee x') \supseteq g_{\mathcal{H}}(x \vee x') = g_{\mathcal{H}}(1) = f_{\mathcal{H}}(1) \tag{1}$$

for all $x \in \mathcal{H}$. This implies that $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$.

□

5 Ultra intersectional soft filter (Ultra IS-filter)

In this section, we introduce the concept of ultra IS-filter and investigate some of the properties. Also we introduce the concept of prime IS-filter and investigate the relation between ultra IS-filter and prime Boolean IS-filter.

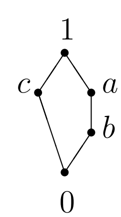
Definition 5.1. A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is called an ultra IS-filter of $\mathcal{H}$ if it is an IS-filter of $\mathcal{H}$ that satisfies:

$$(\forall x \in \mathcal{H}) \ \ (f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1) \ \ or \ \ f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)).$$

We give an example of an ultra IS-filter.

Example 5.2. Let $\mathcal{H} = \{0, a, b, c, 1\}$ be a set with the following Cayley table and Hasse diagram:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	0	1	1	c	1
c	0	a	a	1	1
1	0	a	b	c	1



Then $\mathcal{H}$ is a Heyting algebra.

Let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ in which

$$f_{\mathcal{H}}(x) := \begin{cases} \tau_1 & \text{if } \ x \in \{1, a, b\}, \\ \tau_2 & \text{otherwise}, \end{cases}$$

where $\tau_2 \subsetneq \tau_1 \in \mathcal{H}$. Then $f_{\mathcal{H}}$ is an ultra IS-filter of $\mathcal{H}$.

In the following theorem, we investigate the characterization of ultra IS-filter.

Theorem 5.3. *For an IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$, the following assertions are equivalent:*

(1) *$f_{\mathcal{H}}$ is an ultra IS-filter,*

(2) $(\forall x, y \in \mathcal{H})$ $(f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ *and* $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1) \Rightarrow f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(1)$ *and* $f_{\mathcal{H}}(y \to x) = f_{\mathcal{H}}(1))$.

Proof. Suppose that $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1)$. Then $f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y') = f_{\mathcal{H}}(1)$ by hypothesis. Since

$$f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(x \to 0) = f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$$

we get $f_{\mathcal{H}}(x \to y) \geq f_{\mathcal{H}}(1)$ and so $f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(1)$. Similary, it follows from $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1)$ that $f_{\mathcal{H}}(y \to x) = f_{\mathcal{H}}(1)$.

Conversely, let $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1)$ imply $f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y \to x) = f_{\mathcal{H}}(1))$. Assume that $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$. Since $0 \leq x$, we have $f_{\mathcal{H}}(0) \subseteq f_{\mathcal{H}}(x)$. If $f_{\mathcal{H}}(0) = f_{\mathcal{H}}(1)$ then $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1)$. This is contradiction. So $f_{\mathcal{H}}(x \to 0) = f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is an ultra IS-filter. □

In the following definition, we introduce the concept of prime IS-filter.

Definition 5.4. An IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$ is said to be prime IS-filter if the following assertion is valid.

$$(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(y)) \,.$$

Theorem 5.5. *Every ultra IS-filter is a prime IS-filter.*

Proof. Suppose that $f_{\mathcal{H}}$ is an ultra IS-filter and let $x, y \in \mathcal{H}$. By (hp14), we get $(x \vee y) \leq (x \to y) \to y$. By $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$, we have

$$f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}((x \to y) \to y).$$

From $0 \leq y$ and Proposition 2.2 hp(5), we get $(x \to y) \to y \leq x' \to y$. Thus, $f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}((x \to y) \to y) \subseteq f_{\mathcal{H}}(x' \to y)$ by Definition 3.1.

So

$$f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(x' \to y).$$

For any $x \in \mathcal{H}$, if $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1)$. then

$$f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(y).$$

If $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ then

$$f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$$

by Hypothesis. Thus,

$$\begin{aligned} f_{\mathcal{H}}(y) &\supseteq f_{\mathcal{H}}(x') \cap f_{\mathcal{H}}(x' \to y) \\ &= f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(x' \to y) \\ &= f_{\mathcal{H}}(x' \to y) \end{aligned}$$

by Definition 3.1. Therefore,

$$f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(x' \to y) \subseteq f_{\mathcal{H}}(y) \subseteq f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(y).$$

This means that $f_{\mathcal{H}}$ is a prime IS-filter of $\mathcal{H}$.

□

The converse of Theorem 5.5 is not true in general as can be seen by the following example.

Example 5.6. Let $\mathcal{H} = [0, 1]$ and define $\wedge, \vee$ and implication $\rightarrow$ on $\mathcal{H}$ as follows:

$$\begin{cases} x \wedge y & = \min\{x, y\}, \\ x \vee y & = \max\{x, y\} \end{cases} \quad x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y \end{cases}$$

for all $x, y \in \mathcal{H}$. Then $\mathcal{H}$ is a Heyting-algebra. (In Example 4.3) Let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ in which

$$f_{\mathcal{H}}(x) := \begin{cases} \tau_1 & \text{if } x \in [0, 0.5], \\ \tau_2 & \text{if } x \in (0.5, 1], \end{cases}$$

where $\tau_1 \subsetneq \tau_2$ in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is a prime IS-filter of $\mathcal{H}$. But it is not an ultra IS-filter of $\mathcal{H}$ over U since $f_{\mathcal{H}}(0.5) \neq f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(0.5') \neq f_{\mathcal{H}}(1)$.

We introduce the concept of prime Boolean IS-filter.

Definition 5.7. An IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$ is said to be prime Boolean IS-filter if it is both prime IS-filter and Boolean IS-filter.

In the following theorem, we investigate the relation between ultra IS-filters and prime Boolean IS-filters.

Theorem 5.8. *In a Heyting-algebra $\mathcal{H}$, the notion of an ultra IS-filter coincides with the notion of prime Boolean IS-filter.*

Proof. In Theorem 5.5, we show that every ultra IS-filter is a prime IS-filter. For any $x \in \mathcal{H}$, since $x \leq x \vee x'$, $x' \leq x \vee x'$, we get

$$f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}(x \vee x'),\ f_{\mathcal{H}}(x') \subseteq f_{\mathcal{H}}(x \vee x')$$

According to the definition of ultra IS-filter, we have

$$f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1) \quad or \quad f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1).$$

Thus

$$f_{\mathcal{H}}(1) \subseteq f_{\mathcal{H}}(x \vee x').$$

From this and Definition 3.1(f1), we get

$$f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \vee x').$$

This means that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$.

Conversely, suppose that $f_{\mathcal{H}}$ is a Boolean prime IS-filter of $\mathcal{H}$.

For any $x \in \mathcal{H}$,

$$f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1) \leq f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(x')$$

by Definitions 4.1 and 5.1

Let $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$. Then

$$f_{\mathcal{H}}(x) \leq f_{\mathcal{H}}(1), f_{\mathcal{H}}(x') \leq f_{\mathcal{H}}(1),$$

by Definition 3.1 (f1). So we have $f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$. Thus, $f_{\mathcal{H}}$ is an ultra IS-filter of $\mathcal{H}$.

□

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Filters of Heyting algebras on soft set theory

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(국문초록)

수학에서 Heyting algebras는 Boolean algebras의 일반화로 여겨지는 특별한 bounded lattice입니다. 19세기 Luitzen Brouwer는 수학적 직관주의의 철학을 정립했습니다. 직관주의는 창조적인 생각에 근거한 것이며 직접적인 증거에 의해서만 증명이 될 수 있다고 믿었습니다. Brouwer의 제자인 Arend Heyting은 이러한 개념을 그의 이름을 본떠 정형화 시켰습니다. Heyting algebra는 중요한 역할을 함에 동시에 생물학적 유전적 코드, 다이나믹 시스템, 대수학 등 다양한 방면에 응용이 가능합니다.

경제학, 공학, 환경 등 다양한 분야에서의 불확실한 데이터 모델링의 복잡성은 기존 방법을 정상적으로 사용할 수 없게 만듭니다. 이러한 문제점들을 극복하고자 Molodtsov는 새로운 수학적인 방법인 soft set 이라는 개념을 불확실성을 해결하기위해 도입했습니다. Maji et al. 또한 soft set이라는 개념을 적용하고자 연구하였습니다. 그 이후로 soft set이라는 개념은 경제학, 공학, 환경, 인포메이션 사이언스, 인텔리전스 시스템, 대수적 구조 전반에 걸쳐 다양한 방식으로의 적용이 가능했습니다.

이 논문에는 intersection soft filter (IS-filter), Boolean intersectional soft filter (Boolean IS-filter), ultra intersectional soft filter (ultra IS-filter)등을 정의하고 관계되어있는 특성들을 조사할 것입니다. IS-filter와 Boolean IS-filter의 특성을 논의하고 IS-filter와 Boolean IS-filter 간의 관계를 알아봅니다.

2장에서는 Heyting algebra의 정의와 특성들을 보여줌과 동시에 필터와 soft set에 관해서 설명합니다.

3장에서는 IS-filter의 정의 및 특성에 관한 것입니다.

4장에서는 Boolean IS-filter의 개념을 소개하고 속성 중 일부를 조사하고 IS-filter와 Boolean IS-filter의 관계를 조사합니다.

5장에서는 ultra IS-filter의 개념을 소개하고 일부 속성을 조사합니다. 그리고 prime IS-filter의 개념을 소개하고 ultra IS-filter와 prime Boolean IS-filter의 관계를 조사합니다.